Temperature dependent material properties in incompressible flow

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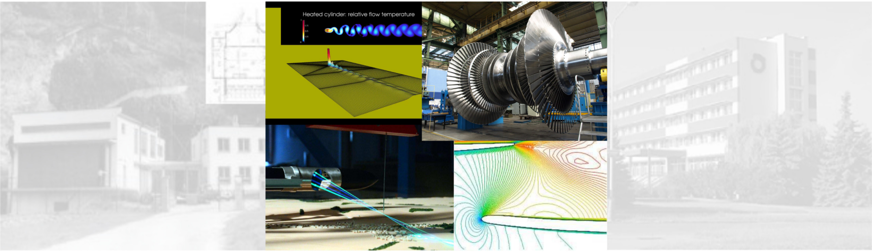
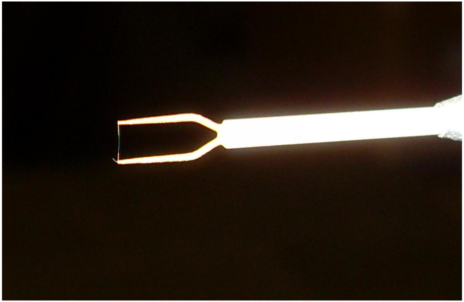
Department of Fluid Dynamics

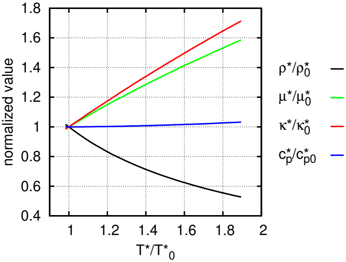
Laboratories of

- Turbulent Shear Flows transition to turbulence, structure and development of turbulent flow
- Internal Flows compressible transonic flow (compressor/turbine blades), vibrating profiles, ejectors, ...
- Environmental Aerodynamics atmospheric boundary layer, atmospheric flows, pollutant dispersion
- Computational Fluid Dynamics



- Simulation of the flow around probes for hot wire anemometry.
- Thin (cca $5\mu m$) heated wire-low Reynolds number ($\mathrm{Re} < 160$).
- Stationary or periodic vortex shedding regime.
- Data from experiments available (setup for parallel vortex shedding-2D simulation should suffice).





Variables with physical units have $*$ in superscript, subscript 0 denotes values belonging to a chosen reference temperature T_0^* (here $T_0^* = 285K$; ρ^*-density; $\rho_0^* = \rho^*(T_0^*)$; μ^*- dynamic viscosity; κ^*- thermal conductivity; c_p^* - spec. heat at constant pressure, variation negligible in this case)

[1] Wang A.-B., Trávníček Z., Chia K.-C.: On the relationship of effective Reynolds number and Strouhal number for the laminar vortex shedding of a heated circular cylinder, Phys Fluids (2000) 12:6, 1401-1410
[2] Trávníček Z., Wang A.-B., Tu W.-Y.: Laminar vortex shedding behind a cooled circular cylinder, Exp Fluids (2014) 55:1679
[3] Maršík F., Trávníček Z., Yen, R.-H., et. al.: Sr-Re-Pr relationship for a heated/cooled cylinder in laminar cross flow. Proceedings of CHT-08 ICHMT International Symposium on Advances in Computational Heat Transfer,2008, Marrakech, Morocco

Coupled evolutionary Incompressible Navier-Stokes-Fourier system

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla\pi+\frac{1}{\mathrm{Re}}\nabla\cdot\left[2\mu\mathbb{D}+\lambda\left(\nabla\cdot\mathbf{v}\right)\mathbb{I}\right]+\mathbf{f}_{\mathbf{v}} \tag{1a}$$

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$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{1}{\mathrm{RePr}} \nabla \cdot \kappa \nabla T + f_T \tag{1c}$$

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..dimensionless formulation in primitive variables, non-conservative form

$\mathbf{v} = (v_1,\ v_2)^{\mathrm{T}}$	velocity
$(\cdot)^{\mathrm{T}},\ \nabla, \nabla\cdot$	matrix vector transposition, gradient, divergence
$\mathbb{D}$, $\mathbb{I}$	$\frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right]$, unit tensor
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$\mathbf{f_v}$	volumetric momentum source, e.g. buoyancy
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$\mathrm{Re} = \frac{L^* \lvert\mathbf{v}^*_\infty\rvert \rho^*_\infty}{\mu^*_\infty}$	Reynolds number (L^* is characteristic length, e.g. cyl. diameter)
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$\mathbf{v} = (\ldots)^{\mathrm{T}}$

$(\cdot)^{\mathrm{T}}$

$\mathbb{D}$, I

t

The stress tensor represents generalized model of Newtonian fluid, but

what is λ and π?

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b)

c)

Physical meaning of "pressure" in considered model

We call *thermodynamic pressure* the variable acting in the equation of state, e.g. $\pi = \rho R T$ for ideal gas. But, instead of (1a), we are going to solve

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \frac{1}{\mathrm{Re}}\nabla\cdot\left[2\mu\mathbb{D} - \frac{2}{3}\mu\left(\nabla\cdot\mathbf{v}\right)\mathbb{I}\right] + \mathbf{f}_{\mathbf{v}} \qquad (2)$$

where $p = \pi - \mu_b \nabla\cdot\mathbf{v}$ is *mean* or *mechanical* pressure, while $\mu_b = \lambda + \frac{2}{3}\mu$ is the *bulk viscosity*. Above equation has the same structure as (1a) while setting $\lambda = -\frac{2}{3}\mu$ (or equivalently $\mu_b = 0$, c.f. Stokes hypothesis), but we avoid specification of μ_b whose precise experimental determination is still open. Our variable p is not thermodynamic pressure.

..dime

$\mathbf{v} =$

$(\cdot)^T$

$\mathbb{D}$, I

t

T

$\rho =$

$\mu = \mu(T)$	dynamic viscosity
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c)

We consider nontrivial mass source m in the continuity equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=m$$

for testing on *manufactured solutions*, but regarding physical meaning of the whole system it can't model the change of mass.

..dime

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$(\cdot)^T$

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Generalized Fourier law is included in the heat equation (1c), which is in form valid for calorically perfect fluid ($e=c_pT$ or $e=c_VT$). A minor change $c_p\to c_V$ and redefinition of Pr switches between gases and liquids, while the equation form is preserved.

..dime

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$(\cdot)^T$

$\mathbb{D}$, $\mathbb{I}$

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Incompressible Navier-Stokes solver

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{v} + \mathbf{f_v} \tag{2a}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2b}$$

Semi-implicit/Projection/Splitting/predictor-corrector/IMEX/velocity-correction/KIO scheme
(Karniadakis, Israeli, Orszag 1991)
Backward difference formula

$$\partial \mathbf{v}/\partial t \approx \frac{\gamma \mathbf{v}_{n+1} - \sum_{q=0}^{Q} \alpha_q \mathbf{v}_{n-q}}{\Delta t} \tag{3}$$

and consistent extrapolation of non-linearities

$$[\mathbf{v} \cdot \nabla \mathbf{v}]_{n+1} \approx \sum_{q=0}^{Q} \beta_q \, [\mathbf{v} \cdot \nabla \mathbf{v}]_{n-q} \equiv [\mathbf{v} \cdot \nabla \mathbf{v}]^* \tag{4}$$

(extrapolated terms are denotes $[]^*$ henceforward)

Decouple the system by application of $\nabla\cdot$ to (2), what results in the pressure-Poisson eq.

$$\nabla^2 p_{n+1} = \nabla \cdot \left(\frac{1}{\Delta t} \sum_{q=0}^{Q} \alpha_q \mathbf{v}_{n-q} - [\mathbf{v} \cdot \nabla \mathbf{v}]^* + \mathbf{f}_{n+1} \right) \tag{5a}$$

denoting $\hat{\mathbf{v}} = \sum_{q=0}^{Q} \alpha_q \mathbf{v}_{n-q} - \Delta t \, [\mathbf{v} \cdot \nabla \mathbf{v}]^*$

$$\nabla^2 p_{n+1} = \nabla \cdot \left(\frac{\hat{\mathbf{v}}}{\Delta t} + \mathbf{f}_{n+1} \right) \tag{5b}$$

supplemented by the Neumann boundary condition

$$\frac{\partial p}{\partial \mathbf{n}} = \left\{ \left[-\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\mathrm{Re}} \nabla \times \nabla \times \mathbf{v} \right]^* + \mathbf{f}_{n+1} \right\} \cdot \mathbf{n} \tag{5c}$$

Having ∇p_{n+1} we can solve the velocity-correction step for $\mathbf{v}_{n+1}$

$$\nabla^2 \mathbf{v}_{n+1} - \frac{\gamma \mathrm{Re}}{\Delta t} \mathbf{v}_{n+1} = \mathrm{Re} \left(\nabla p_{n+1} - \frac{\hat{\mathbf{v}}}{\Delta t} - \mathbf{f}_{n+1} \right) \tag{5d}$$

Acceleration term in pressure Neumann BC is approximated from BDF (known a priori for Dirichlet BC)

$$\left[\frac{\partial \mathbf{v}}{\partial t}\right]\Big|_{n+1} \approx \frac{\gamma \mathbf{v}_{n+1} - \sum_{q=0}^{Q} \mathbf{v}_{n-q}}{\Delta t} \tag{6}$$

Testing (in 2D): known smooth functions, $\nabla \cdot \mathbf{v} = 0$

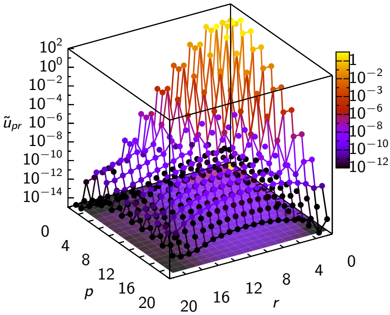
$$\mathbf{v}(\mathbf{x}, t) = \begin{pmatrix} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{pmatrix} = \begin{pmatrix} 2\cos(\pi x)\cos(\pi y)\sin(t) \\ 2\sin(\pi x)\sin(\pi y)\sin(t) \end{pmatrix} \tag{7a}$$

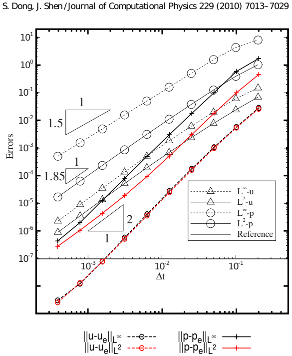
$$p(\mathbf{x}, t) = 2\sin(\pi x)\sin(\pi y)\cos(t) \tag{7b}$$

$$\mathbf{f}(\mathbf{x}, t) = \begin{pmatrix} 2\cos(\pi x)\cos(\pi y)\cos(t) - 4\pi\cos\ldots \\ 2\sin(\pi x)\sin(\pi y)\cos(t) + 4\pi\cos\ldots \end{pmatrix} \tag{8}$$

Convergence properties

Initialisation of IMEX2 using IMEX1 scheme with $\Delta t_{I1} < \Delta t_{I2}$ $\Omega = [0:2] \times [-1:1]$ (two quadrilateral elements), $NM = 15$, $t \in [0:1]$, $\mathrm{Re} = 1$





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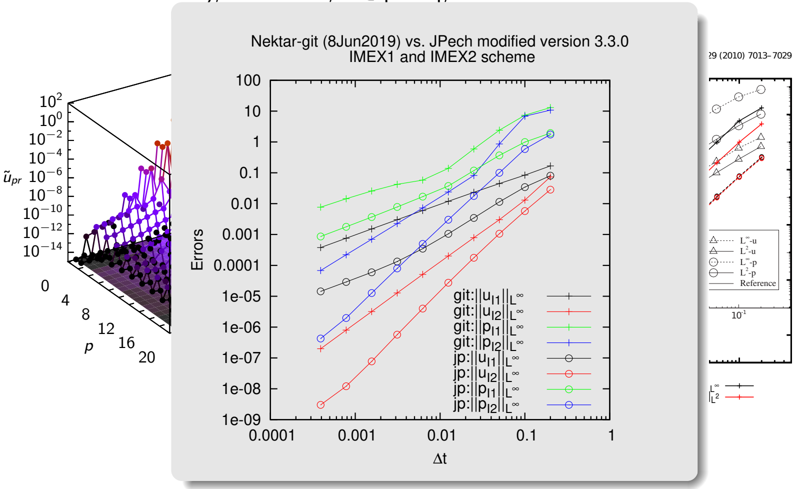

Intermediate steps

1. Homogeneous divergence
2. Divergence as a given function
   - velocity-correction for $\nabla \cdot \mathbf{v} = m(\mathbf{x}, t)$
   - extension of stress tensor $\nabla^2 \mathbf{v} \to \nabla \cdot \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\right] - \frac{2}{3} \nabla \nabla \cdot \mathbf{v}$
   - Advection-diffusion for temperature (one direction of coupling, velocity independent of temperature)
   - $\mu = \mu(T)$ For $\nabla \cdot \mathbf{v} = 0$ studied in Karamanos and Sherwin: Applied Numerical Mathematics 33 (2000) 455–462 (IDEA: $\mu(T(\mathbf{x}, t)) = \bar{\mu} + \mu_s(t, \mathbf{x})$, where $\bar{\mu} = \mathrm{const.}$ or $\bar{\mu} = \bar{\mu}(\mathbf{x})$)
   - $\kappa(T) = \bar{\kappa} + \kappa_s(\mathbf{v}, t)$
3. Divergence estimated from continuity equation, since $\rho = \rho(T)$ & $\rho \neq \rho(p)$
   - in Navier-Stokes
   - in energy balance

Demonstration of variable property splitting: Nonlinear energy equation

$\kappa = \kappa(T)$

Tabulated data, e.g. power function fit (no a priori restriction to the function form):

$\kappa^*(T^*) = \kappa_0^* \left(\frac{T^*}{T_0^*}\right)^{\omega_\kappa} \Rightarrow \kappa(T) = \frac{\kappa_0^*}{\kappa_\infty^*} \left(T \frac{T_\infty^*}{T_0^*}\right)^{\omega_\kappa}$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \frac{1}{\mathrm{RePr}} \nabla \cdot \kappa \nabla T + f_T \quad (9)$$

$\kappa(T) = \bar{\kappa} + \kappa_s$ ($\bar{\kappa} = \mathrm{const.}$ or $\bar{\kappa} = \kappa(\mathbf{x})$ and $\kappa_s = \kappa_s(\mathbf{x}, t)$)

$$\bar{\kappa} \nabla^2 T_{n+1} - \frac{\gamma \mathrm{RePr}}{\Delta t} T_{n+1} = \mathrm{RePr} \left(-\frac{\hat{T}}{\Delta t} - f_{T\,n+1} \right) - [\nabla \cdot \kappa_s \nabla T]^* \quad (10)$$

where $\hat{T} = \sum_{q=0}^{Q} \alpha_q T_{n-q} - \Delta t \, [\mathbf{v} \cdot \nabla T]^*$. If $\bar{\kappa} = \bar{\kappa}(\mathbf{x})$ the formulation complicates.

Temperature dependent properties in Navier-Stokes & Full continuity equation

- $\mu(T) = \frac{\mu_0^*}{\mu_\infty^*}\left(T\frac{T_\infty^*}{T_0^*}\right)^{\omega_\mu}, \quad \rho(T) = \frac{\rho_0^*}{\rho_\infty^*}\left(T\frac{T_\infty^*}{T_0^*}\right)^{\omega_\rho}$
- $\nabla \cdot \mathbf{v}$ follows from the continuity equation
- The "mass source" m is included for the testing only

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \frac{1}{\mathrm{Re}}\left\{\nabla\cdot\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right] - \frac{2}{3}\nabla\left(\mu\nabla\cdot\mathbf{v}\right)\right\} + \mathbf{f}_{\mathbf{v}} \quad (11)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = m \quad (12)$$

Forward estimate of velocity divergence

We need to evaluate $\nabla \cdot \mathbf{v}$ for both the pressure-Poisson and velocity-correction steps

$$[\nabla\cdot\mathbf{v}]_{n+1} \approx \frac{1}{[\rho]^*}\left\{m_{n+1} - [\mathbf{v}\cdot\nabla\rho]^* - \left[\frac{\partial\rho}{\partial t}\right]^{**}\right\} \quad (13)$$

where $[\,]^{**}$ denotes extrapolation of the approximation of density time derivative

$$\left[\left.\frac{\partial\rho}{\partial t}\right|_n\right]^* \approx \left[\frac{\gamma\rho_n - \sum_{q=0}^{Q}\alpha_q\rho_{n-1-q}}{\Delta t}\right]^* \equiv \left[\frac{\partial\rho}{\partial t}\right]^{**} \quad (14)$$

Pressure-Poisson step

Neumann pressure boundary condition

$$\frac{\partial p}{\partial \mathbf{n}} = \mathbf{n}\cdot \left\{ \left[-\rho \frac{\partial \mathbf{v}}{\partial t} - \rho \mathbf{v}\cdot\nabla\mathbf{v} + \frac{1}{\mathrm{Re}} \left(-\mu \nabla\times\nabla\times\mathbf{v} + \nabla\mu\cdot\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right]\right)\right]^{*} + \frac{1}{\mathrm{Re}}\left(-\frac{2}{3}\left[\nabla\mu\right]^{*}\left[\nabla\cdot\mathbf{v}\right]_{n+1} + \frac{4}{3}[\mu]^{*}\nabla\left[\nabla\cdot\mathbf{v}\right]_{n+1}\right) + \mathbf{f}_{n+1}\right\} \tag{15}$$

Pressure-Poisson equation

$$\nabla^2 p = \frac{\gamma}{\Delta t}\left(\left[\frac{\partial \rho}{\partial t}\right]^{**} - m_{n+1}\right) + \nabla\cdot\left\{ \frac{\gamma}{\Delta t}[\rho]^{*}\hat{\mathbf{v}} + \frac{1}{\mathrm{Re}}\left[\nabla\mu\cdot\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right) - \nabla\mu\cdot(\nabla\times\nabla\times\mathbf{v})\right]^{*} + \frac{1}{\mathrm{Re}}\left(-\frac{2}{3}[\nabla\mu]^{*}[\nabla\cdot\mathbf{v}]_{n+1} + \frac{4}{3}[\mu]^{*}\nabla[\nabla\cdot\mathbf{v}]_{n+1}\right) + \mathbf{f}_{n+1}\right\} \tag{16}$$

Velocity-correction step

Density split

$$\frac{1}{\rho} = \overline{\left(\frac{1}{\rho}\right)} + \left(\frac{1}{\rho}\right)_s$$

$$\begin{aligned}
&\nabla^2 \mathbf{v}_{n+1} - \frac{\gamma}{\Delta t}\frac{\bar{\rho}}{\bar{\mu}}\mathrm{Re}\mathbf{v}_{n+1} = \\
&\frac{\bar{\rho}}{\bar{\mu}}\left\{-\mathrm{Re}\frac{\gamma}{\Delta t}\hat{\mathbf{v}} + \frac{1}{[\rho]^*}\left[\mathrm{Re}(\nabla p_{n+1} - \mathbf{f}_{n+1}) - \left[\nabla\mu\cdot[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}]\right]^*\right.\right. \\
&\left.\left. + \frac{2}{3}[\nabla\mu]^*[\nabla\cdot\mathbf{v}]_{n+1} - \frac{1}{3}[\mu]^*\nabla[\nabla\cdot\mathbf{v}]_{n+1}\right]\right\} \\
&- (\nabla[\nabla\cdot\mathbf{v}]_{n+1} - [\nabla\times\nabla\times\mathbf{v}]^*)\left[\frac{\bar{\rho}}{\bar{\mu}}\left(\frac{1}{[\rho]^*}\right)_s[\mu]^* + \frac{1}{\bar{\mu}}[\mu]^*_s\right]
\end{aligned} \tag{17}$$

..the last term in square brackets simplifies to $\left[\frac{[\mu]^*}{[\rho]^*} - 1\right]$ if $\overline{\left(\frac{1}{\rho}\right)} = \bar{\mu} = 1$

Convergence for manufactured solution

- $\Omega = [0, 2] \times [-1, 1]$ two elements, $NM = 15$
- Boundary conditions:
  - Dirichlet for $\mathbf{v}$
  - HOPBC (Neumann) for pressure
  - Dirichlet for T
- $\mathrm{Re} = \mathrm{Pr} = 1$
- $$\begin{pmatrix} u \\ v \\ T \\ p \end{pmatrix} = \begin{pmatrix} 2\cos(\pi x)\cos(\pi y)\sin(t) \\ \sin(\pi x)\sin(\pi y)\sin(t) \\ \sin(x)\sin(y)\cos(t) \\ 2\sin(\pi x)\sin(\pi y)\cos(t) \end{pmatrix}$$
- $\mu(T) = \kappa(T) = \rho(T) = (1 + 0.1T)^2$
- veeery long forcing terms defining $\mathbf{f}_\mathbf{v}$ and f_T (snippet follows):

```
<FUNCTION NAME="BodyForce">
           <E VAR="u" VALUE="
           (1.0 + 0.1*sin(x)*sin(y)*cos(t))^(2.0)*(
            2.0*cos(PI*x)*cos(PI*y)*cos(t)
            -2.0*PI*sin(t)*sin(t)*sin(2.0*PI*x)*( 1.0-0.5*sin(PI*y)*sin(PI*y) )
            )
            +2.0*PI*cos(PI*x)*sin(PI*y)*cos(t)
            + (
            0.1*2.0*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0-1.0)*PI*cos(t)*sin(t)*( 4.0*cos(x)*sin(y)*sin(PI*x)*cos(PI*y) + sin(x)*cos(y)
            +4.0*PI*PI*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0)*cos(PI*x)*cos(PI*y)*sin(t)
            +PI*PI*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0)*cos(PI*x)*cos(PI*y)*sin(t) / 3.0
            -2.0*PI*0.1*2.0*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0-1.0)*cos(x)*sin(y)*sin(PI*x)*cos(PI*y)*cos(t)*sin(t)/ 3.0
              )/Re
           -(1.0 + 0.1*sin(x)*sin(y)*cos(t))^(2.0)*gx*g_inf*Char_Length/V_inf*V_inf
            "/>
           <E VAR="v" VALUE="
           (1.0 + 0.1*sin(x)*sin(y)*cos(t))^(2.0)*(
            sin(PI*x)*sin(PI*y)*cos(t)
            +PI*sin(t)*sin(t)*sin(2.0*PI*y)*( 1.0-0.5*sin(PI*x)*sin(PI*x) )
```

Convergence for manufactured solution

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             +4.0*PI*PI*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0)*cos(PI*x)
             +PI*PI*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0)*cos(PI*x)*cos
             -2.0*PI*0.1*2.0*(1.0+0.1*sin(x)*sin(y)*cos(t))^(2.0-1.0)*
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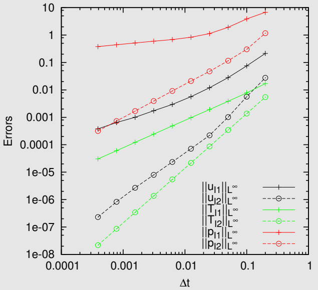
```

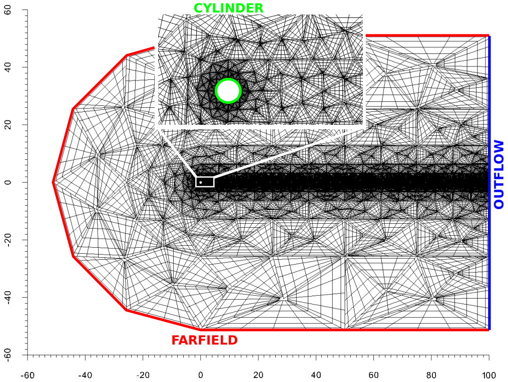
New scheme convergence: manufactured solution



Simulation setup

Triangular mesh with 2360 elements, $NM = 7$, cylinder boundary curve 15 GLL points
Boundary conditions:

- Farfield/Inlet - Dirichlet velocity, HOPBC, Dirichlet temperature
- Cylinder - velocity no-slip, HOPBC, Dirichlet temperature
- Outflow - Zero Neumann for velocity, homogeneous Dirichlet pressure, Zero Neumann for temperature



Cylinder flow influenced by heating/cooling - full model

- buoyancy:
  - $\mathbf{f_v} = \frac{1}{\mathrm{Fr}^2}\,[\rho(T)]^*\,\mathbf{g}$
  - Dirichlet pressure boundary condition

    gradient in direction of gravity force≡shifted hydrostatic pressure: $p|_{\Omega_{OUT}} = \frac{g^*_\infty L^*}{|\mathbf{v}^*_\infty|^2}\,y$
  - Solution to this model gives an estimate of the absolute pressure variations in the wake
- viscous heating:
  - $f_T = \frac{\mathrm{Ec}}{\mathrm{Re}}\left[\mu(T)\left\{2\left[\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\frac{\partial v_2}{\partial x_2}\right)^2\right] + \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)^2\right\}\right]^*$

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$\mathrm{Fr} = \frac{|\mathbf{v}^*_\infty|}{\sqrt{g^*_\infty L^*}}$	Froude number
$\mathbf{g}$	gravity vector (normalized by $g^*_\infty = 9.81 ms^{-1}$)
$\mathrm{Ec} = \frac{|\mathbf{v}^*_\infty|^2}{c^*_p T^*_\infty}$	Eckert number (in case $T = T^* / T^*_\infty$)
$\tilde{T} = \frac{T_W}{T_\infty}$	

Cylinder flow influenced by heating/cooling - full model

- bu…
- vis…

$\Omega_{OUT} = \frac{g_\infty^* L^*}{|\mathbf{v}_\infty^*|^2} y$

…ations in the wake

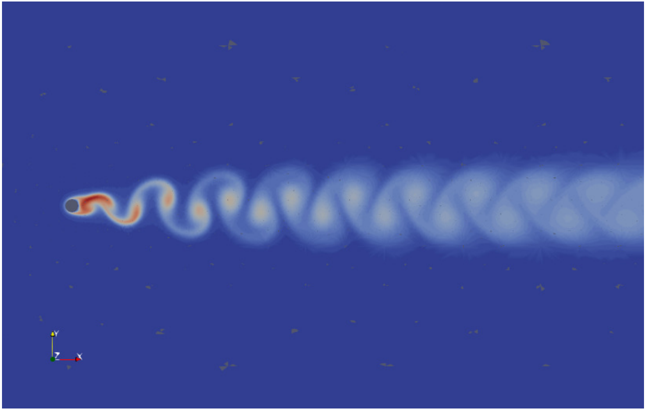
$\mathrm{Fr} = \frac{|}{\sqrt{}}$

$\mathbf{g}$

$\mathrm{Ec} = \frac{|\mathbf{v}}{c_p^*}$

$\tilde{T} = \frac{T_W}{T_\infty}$

Accuracy demonstration: $\tilde{T} \to T_\infty = T_W$, i.e. viscous heating only
$\mathrm{Re} = 121.8$, T variation $\approx 10^{-7}$, ($T \in [0.99999996, 1.000004]$)



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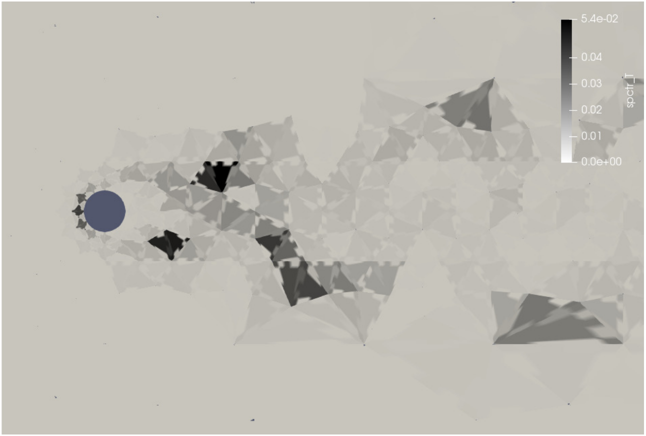
Indication of spatial resolution of temperature field using highest modes in spectra: $\mathrm{Re} = 121.2$, $\tilde{T} = 1.5$, $\frac{1}{\mathrm{Fr}^2} = 0.0026$, $\frac{\mathrm{Ec}}{\mathrm{Re}} = 1e^{-7}$



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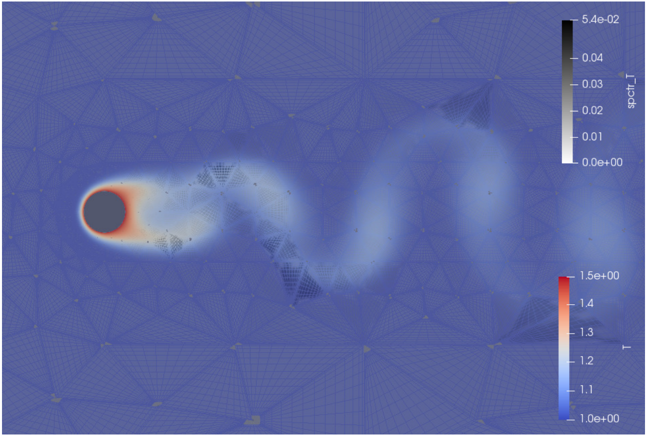

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g

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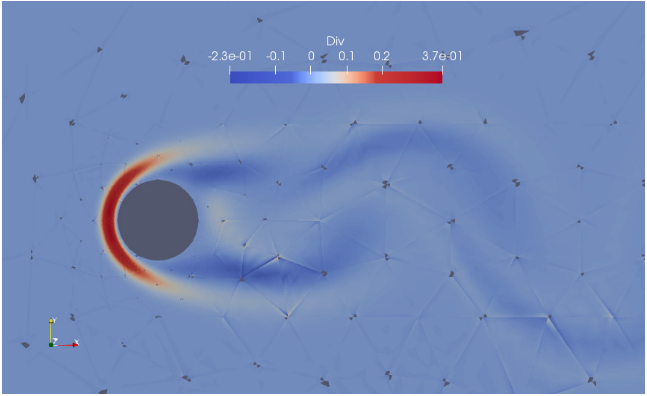
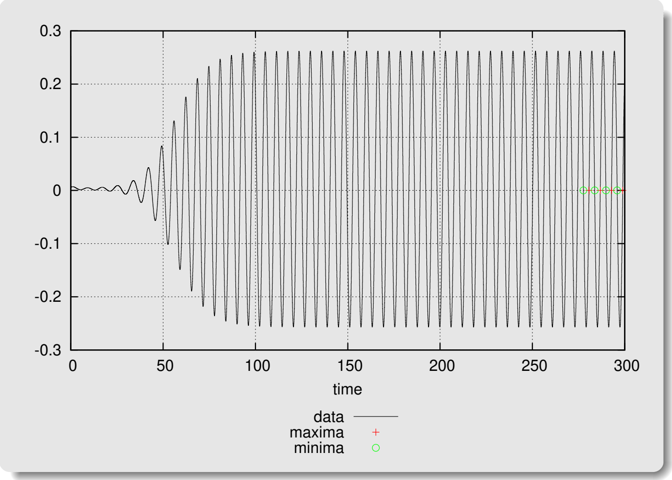
Velocity divergence (without projection to C^0)



Results for heated cylinder flow: frequency of vortex shedding behind heated/cooled cylinder $\mathrm{St} = \mathrm{St}(\mathrm{Re}, T)$ (*St*...Strouhal number)



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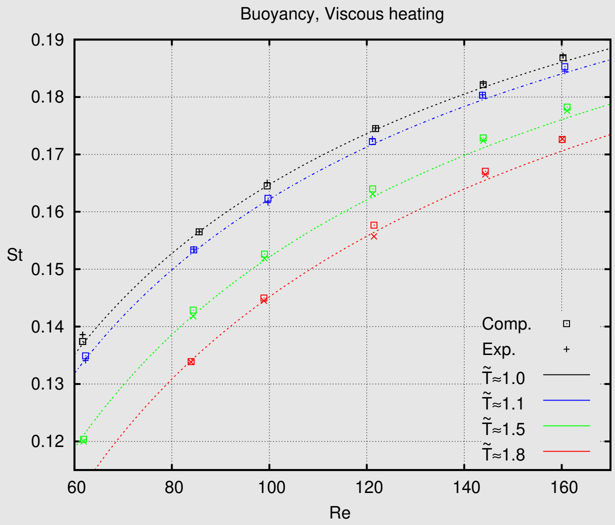

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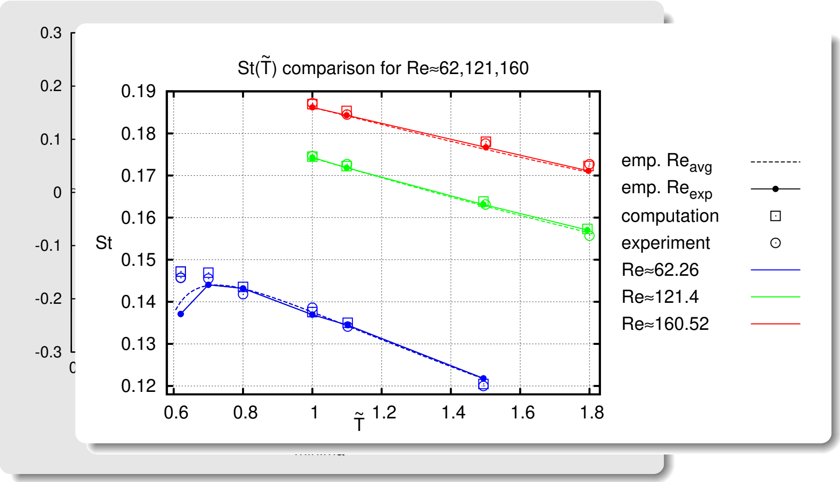

Thank you for attention