Under-resolved DNS of non-trivial turbulent boundary layers via spectral/hp continuous Galerkin methods

Rodrigo C. Moura, Joaquim Peiró and Spencer Sherwin

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(U)RANS is less reliable for “non-standard” flows

Wall-modeled LES still unreliable for certain flows [*]

Wall-resolved LES too expensive at high Reynolds

Wall-resolved implicit LES / under-resolved DNS via spectral/hp methods are promising strategies (CG, DG, etc)

They seem to offer reliable results for highly complex flows at a reduced cost (being affordable at moderate Reynolds)
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Presentation outline

• What are spectral element methods (SEM)?

• What is the SEM-based iLES / uDNS approach?

• How to understand its properties and capabilities?

• For DG, dispersion-diffusion eigenanalysis has proved insightful

• Ongoing eigenanalysis of CG with stabilisation based on SVV (and others)
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- Ongoing eigenanalysis of CG with stabilisation based on SVV (and others)
- **Current work focuses on CG’s performance for a specific test case (non-trivial turbulent BL case)**
Spectral element methods

Extensions of FE / FV via polynomial basis

Kind of spectral method within the element

Natural high-order with geometrical flexibility

Strong / weak inter-element continuity (CG / DG)
**Stabilisation for iLES / uDNS**

Stabilization via SVV or upwind (Riemann) fluxes

Hyperviscous-like dissipation at higher orders

Improved resolution power per DOF employed

Rationale for implicit LES / under-resolved DNS

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \mu \frac{\partial}{\partial x} \left( Q \star \frac{\partial u}{\partial x} \right) \approx
\]

\[
\approx -\mu \sum_k k^2 \hat{Q}_k \hat{u}_k \exp(ikx)
\]
Eigenanalysis for DG – linear advection in 1D
Tests with Burgers turbulence (1D)

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = \frac{A_F}{\sqrt{\Delta t}} \sum_{N \in \mathbb{N}_F} \frac{\sigma_N(t)}{\sqrt{|N|}} \exp \left( i \frac{2\pi N}{L} x \right)
\]

\[
p = 1, \ n_{el} = 2048 \quad \text{and} \quad p = 7, \ n_{el} = 512
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Tests with the inviscid TGV flow
3D energy spectra – adapted 1% rule (3D version)
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Eigenanalysis for CG — linear advection plus SVV

Numerical Dispersion (CG)

Numerical Diffusion (CG)

P=2 to P=8
Eigenanalysis for CG – linear advection plus SVV

**Numerical Dispersion (CG)**

- Real \( (k^* h) / P \)
- k h / P
- P=2 to P=8

**Numerical Diffusion (CG)**

- Imag \( (k^* h) / P \)
- k h / P
- P=2 to P=8
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• Current work focuses on CG’s performance for a specific test case (non-trivial turbulent BL case)
Test case description

- Rotating turbulent BL case proposed by Spalart [*]
- Admits an asymptotic solution (stat. steady in the rotating frame)
- Rich physics, enhanced unsteadiness and stronger anisotropy
- Misalignment between mean-flow shear and Reynolds stresses
- Requires sophisticated modeling and relatively fine grids

\[
\begin{align*}
    u(x, z, t) &= V_o \cos \phi \\
    w(x, z, t) &= V_o \sin \phi \\
    \phi &= ft
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Relevant quantities and available DNSs

- Known laminar solution gives $\delta_L = (2 \nu / f)^{1/2}$

- Several Reynolds can be defined: $R_L$, $R_T$ ... $\text{Re} = \delta_T V_O / \nu$

- Summary of relevant data from available DNS results:

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<th>$R_\ell$</th>
<th>$R_T$</th>
<th>$Re$</th>
<th>$\delta_T/\delta_\ell$</th>
<th>$V_*/V_O$</th>
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<th>$\alpha (^{\circ})$</th>
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<td>500</td>
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<td>7632</td>
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Available solutions at $R_L = 767$

- In previous LESs, the Mixed Lagrangian (ML) model gave best results [*]

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(first node at $y^+ = 1$ and 10 equispaced nodes within $y^+ < 10$, then a constant geometric stretching along the remainder of the BL thickness)
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- **Discretisations employed in the present work:**
  - Run 32p1: 8 equispaced $P = 1$ elements within $y^+ < 10$ and the remaining 24 elements stretched until $y = 1.6 \delta_T$ ... (total $32^3$ dofs)
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  - Run **8p4**: 2 equispaced $P = 4$ elements within $y^+ < 10$ and the remaining 6 elements stretched until $y = 1.6\delta_T$ … (total $32^3$ dofs)
Numerical set-up

- Nektar++ incompressible solver [*]
- Spectral/hp CG with P=1 and P=4
- Using polynomial dealiasing, but no SVV
- Perturbed laminar solution as IC
- Transient takes ~ 10 cycles
- Statistics gathered over ~ 1 cycle
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Results of run 32p1

CG-uDNS (p=1) on $32^3$ dofs VS. dynamic Smagorinsky (FD-based) on $64^3$ dofs

$\hat{u}^+ = \frac{2.44 \ln (y^+)}{V_*} + 5.0$

$y = \delta_r$
Results of run 8p4

CG-uDNS (p=4) on 32^3 dofs VS. mixed Lagrangian model (FD-based) on 64^3 dofs

\[ \hat{u}^+ = 2.44 \ln(y^+) + 5.0 \]

\[ y = \delta_\tau \]
Results of run 8p4

CG-uDNS \((p=4)\) on \(32^3\) dofs VS. mixed Lagrangian model (FD-based) on \(64^3\) dofs
Results of run 8p4

CG-uDNS (p=4) on $32^3$ dofs VS. mixed Lagrangian model (FD-based) on $64^3$ dofs
Why so much success?

- Superior resolution power per DOF at higher orders
- Absence of (often restrictive) modeling assumptions
- Reynolds not too high yet (no SVV need so far)
Future directions

- Analysis of the energy spectrum, and other statistics
- Try and correlate with dispersion-diffusion analysis
- Increase the Reynolds number (and probably add SVV)
Questions