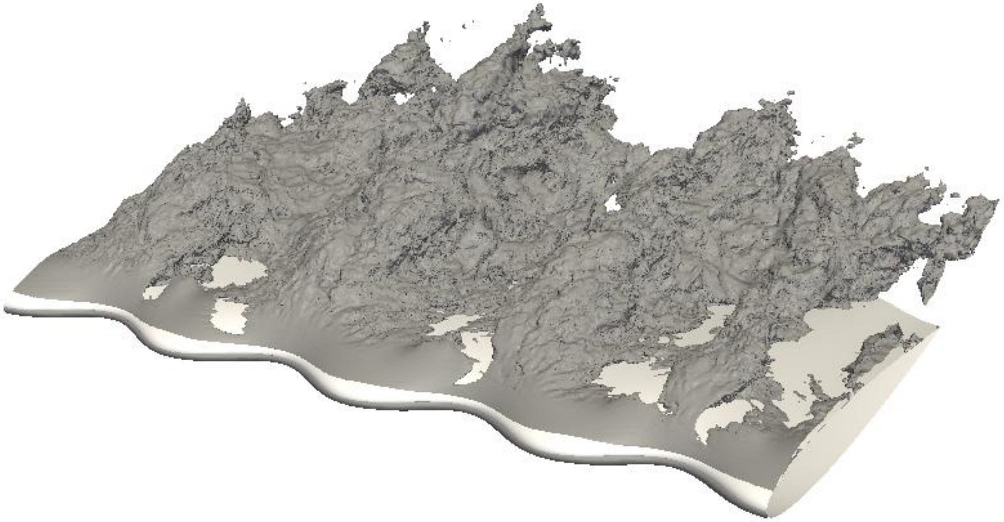
Accuracy and robustness of CG/DG for spatially developing under-resolved turbulent flows



Rodrigo C. Moura
PhD student at Imperial

Nektar++ Workshop 2017
June 14th

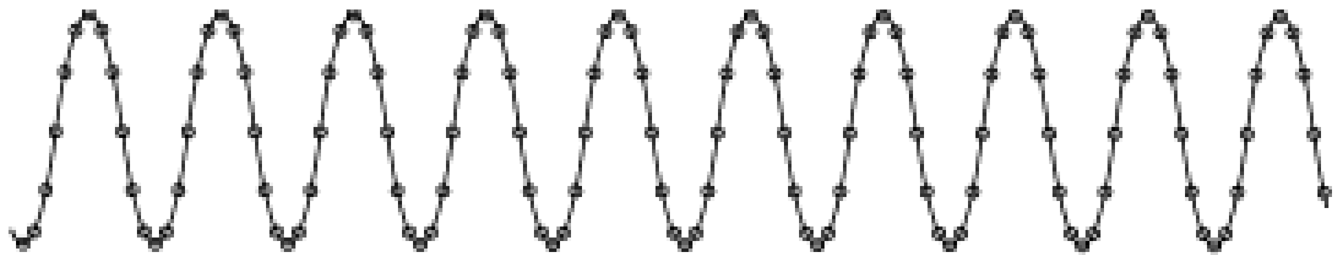
Imperial College London

Introduction & outline

- Eigensolution analysis framework
- Temporal vs. spatial approaches
- DG's behaviour for
  - spatially evolving problems
  - varying upwinding effects
- CG's behaviour for
  - spatially evolving problems
  - robust stabilisation via SVV
- Guidelines for under-resolved simulations
  (including SEM-based iLES / uDNS)

Wave resolution and eigenanalysis

$k\lambda = 2\pi$



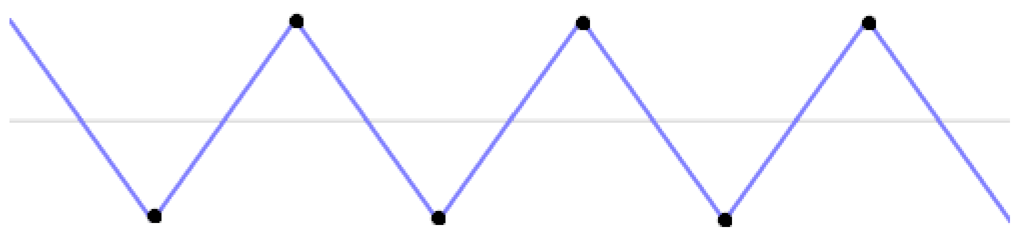
Adequately Sampled Signal

$k h = ?$

$k h / m = ? ?$

Aliased Signal Due to Undersampling

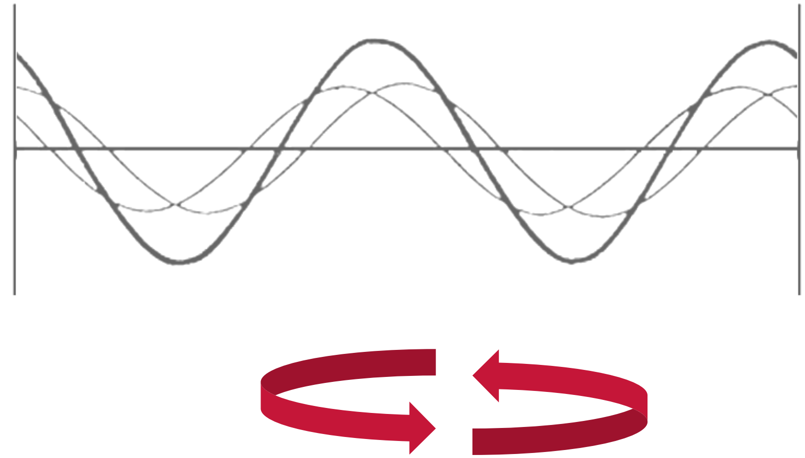
where
$m = p$ for CG
$m = p + 1$ for DG



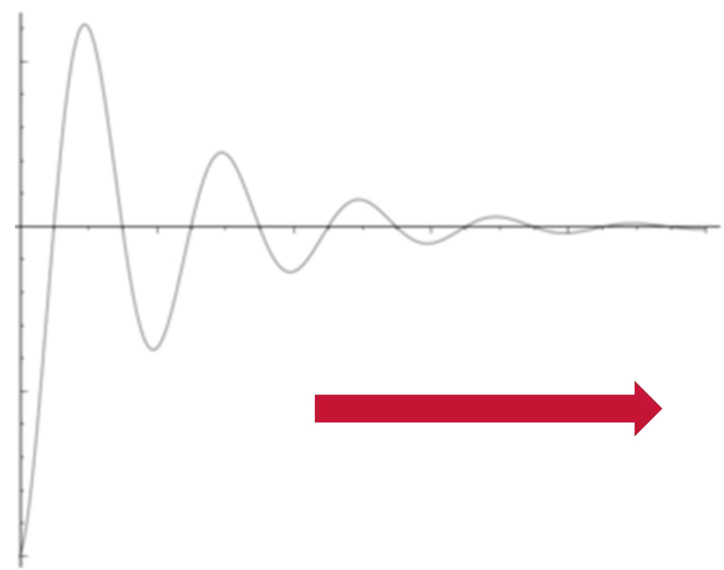
$k h = \pi$

Temporal Vs. Spatial analysis frameworks

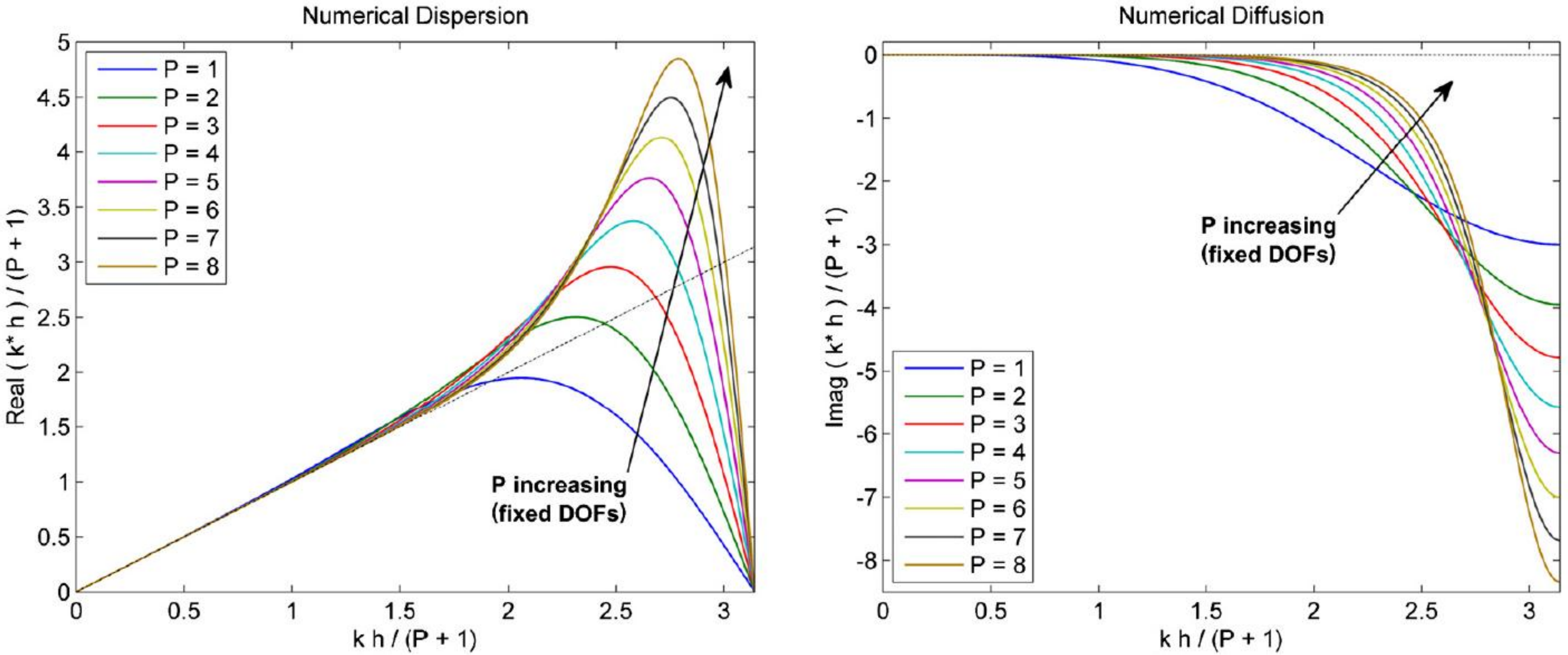
- Temporal approach
  - assumes periodic BCs
  - waves are coherent in space
  - input over all *x* at one *t* (initial condition)
  - driving parameter is the wavenumber κ
  - solution recycled
  - useful for temporally evolving problems



- Spatial approach
  - assumes inflow/outflow type BCs
  - waves are coherent in time
  - input over all *t* at one *x* (inflow boundary)
  - driving parameter is the frequency ω
  - solution renewed
  - useful for spatially developing problems



Temporal eigenanalysis of DG – linear advection in 1D



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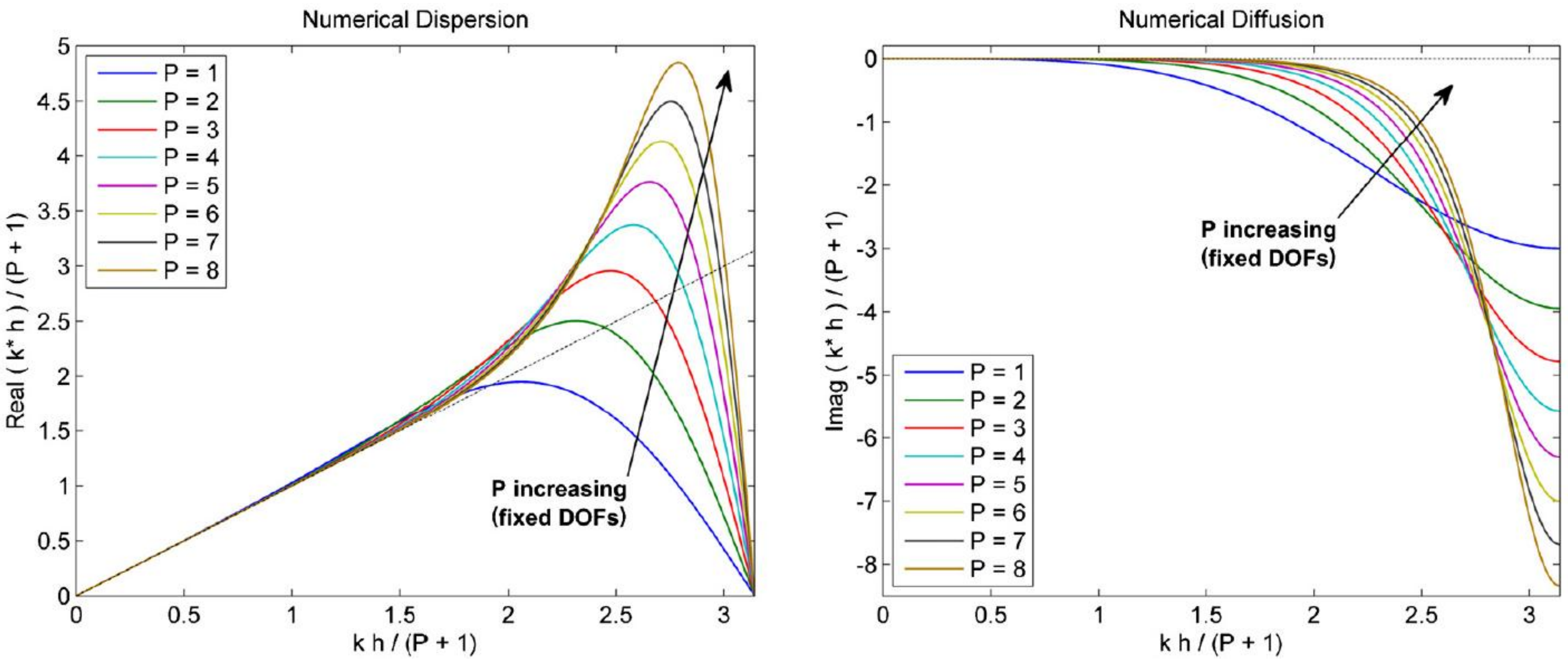

(standard upwinding)

Temporal eigenanalysis of DG – linear advection in 1D

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  (dispersion remains approx. the same, dissipation goes to zero)

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$$\widetilde{\mathrm{F}}_i = \overline{\mathrm{F}}_i - \beta \frac{|\lambda|}{2} \delta q \qquad \beta = \frac{u+c}{u} = 1 + \frac{1}{\mathrm{Mach}}$$

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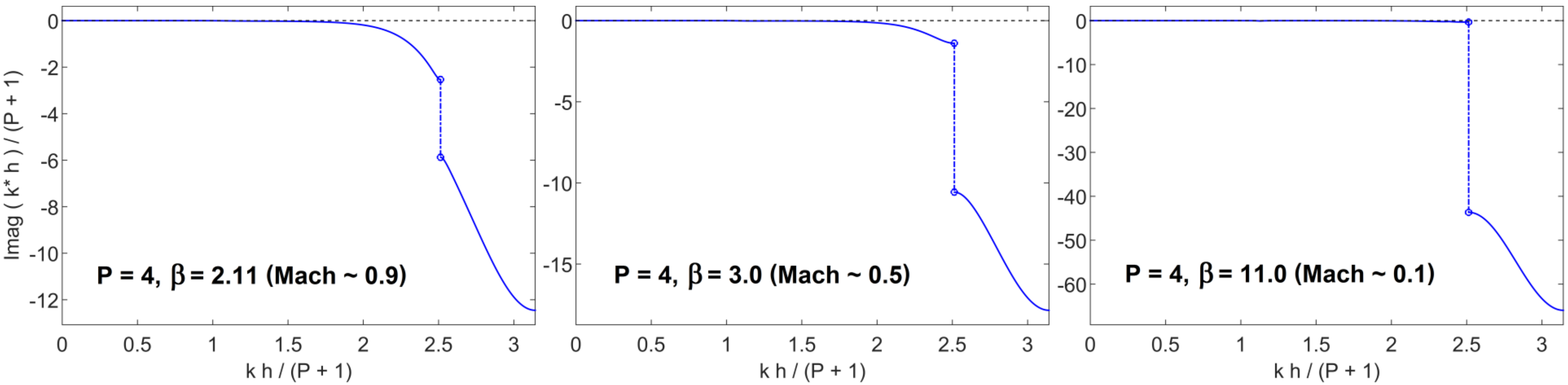

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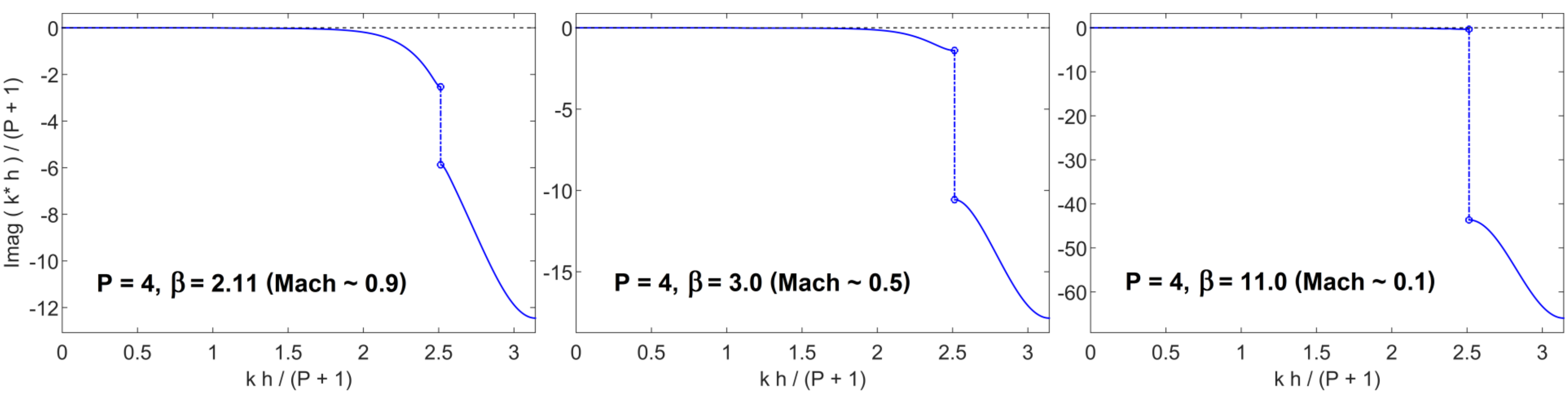

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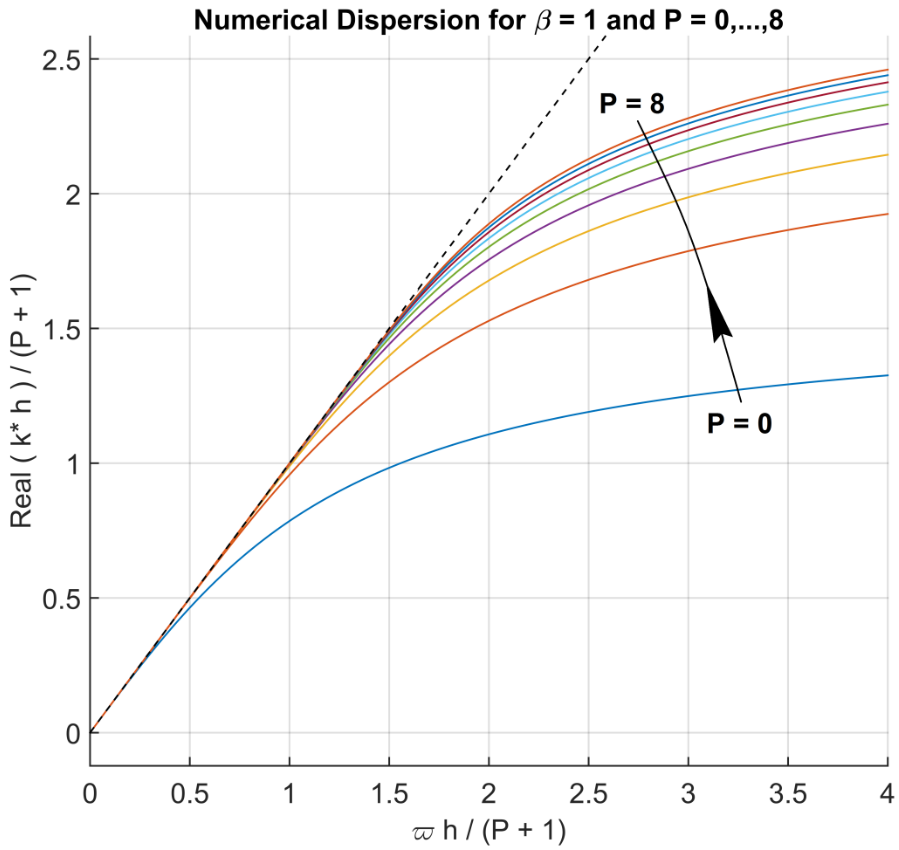
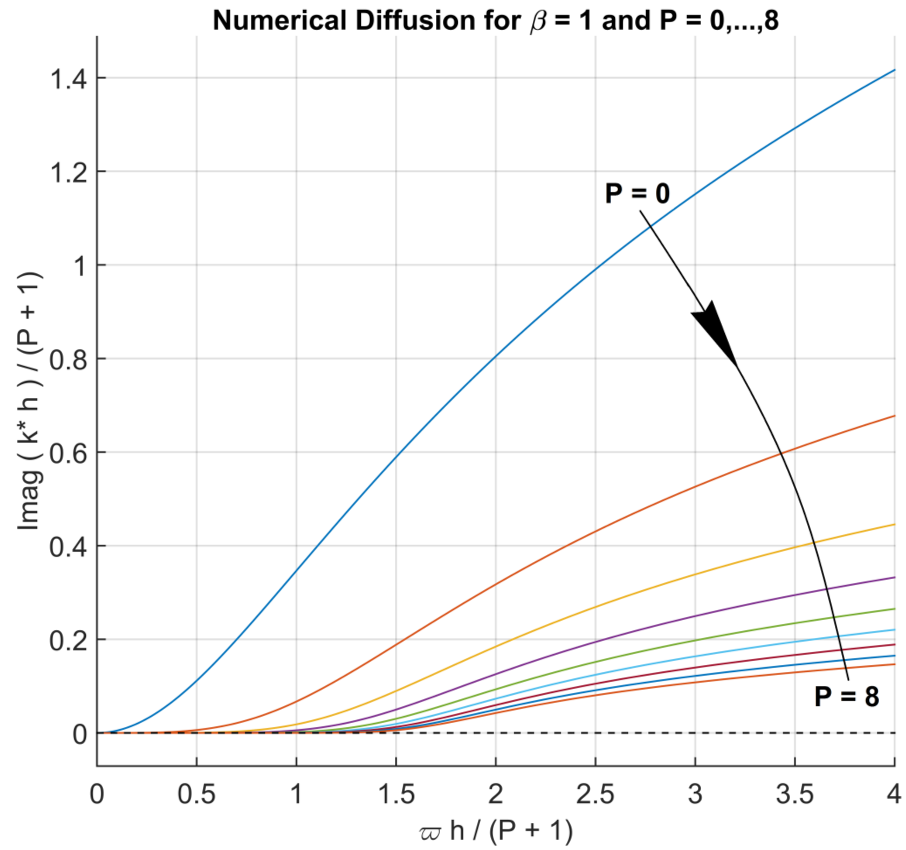
Spatial eigenanalysis of DG – linear advection in 1D





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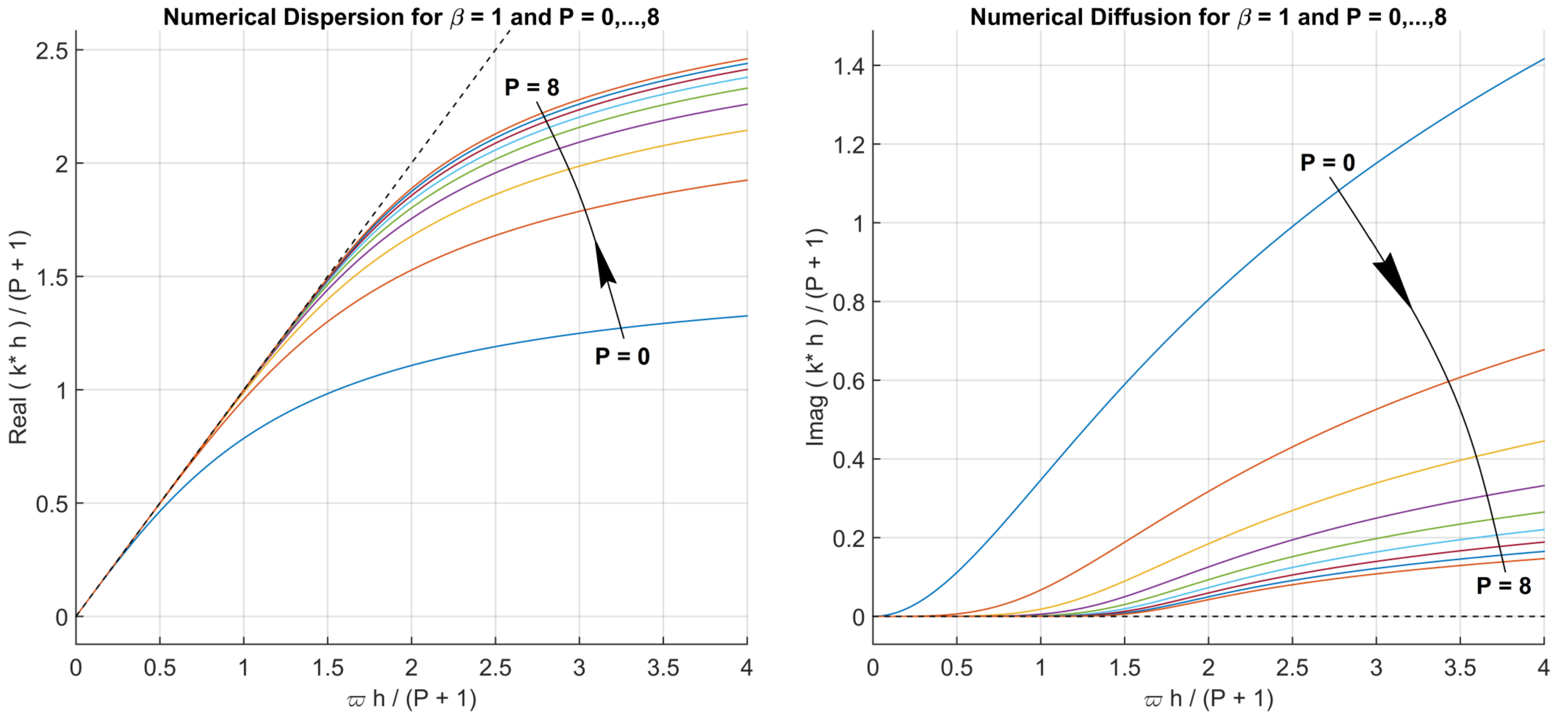

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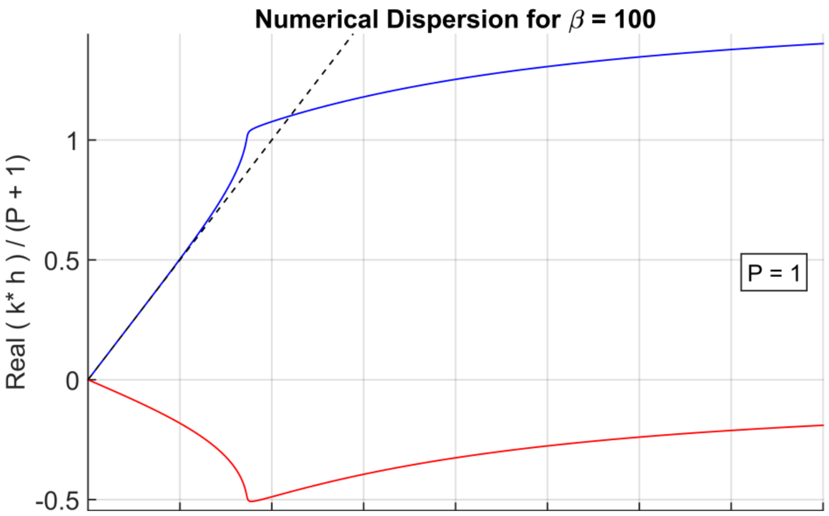

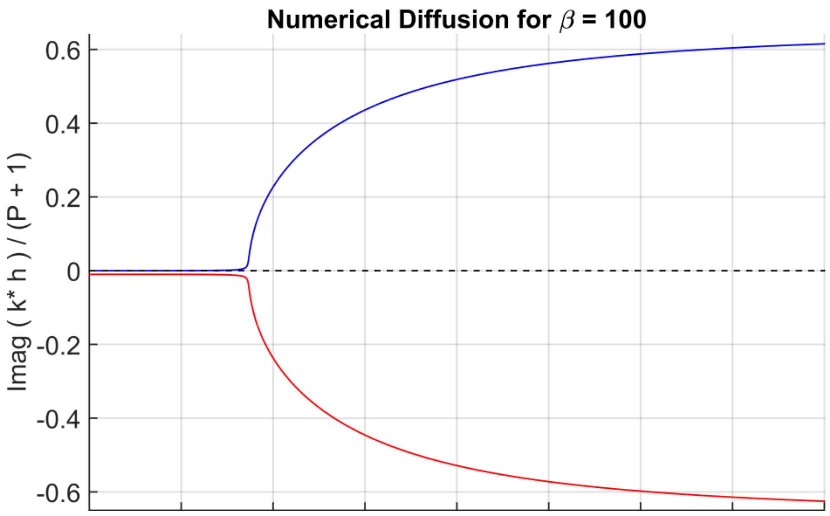

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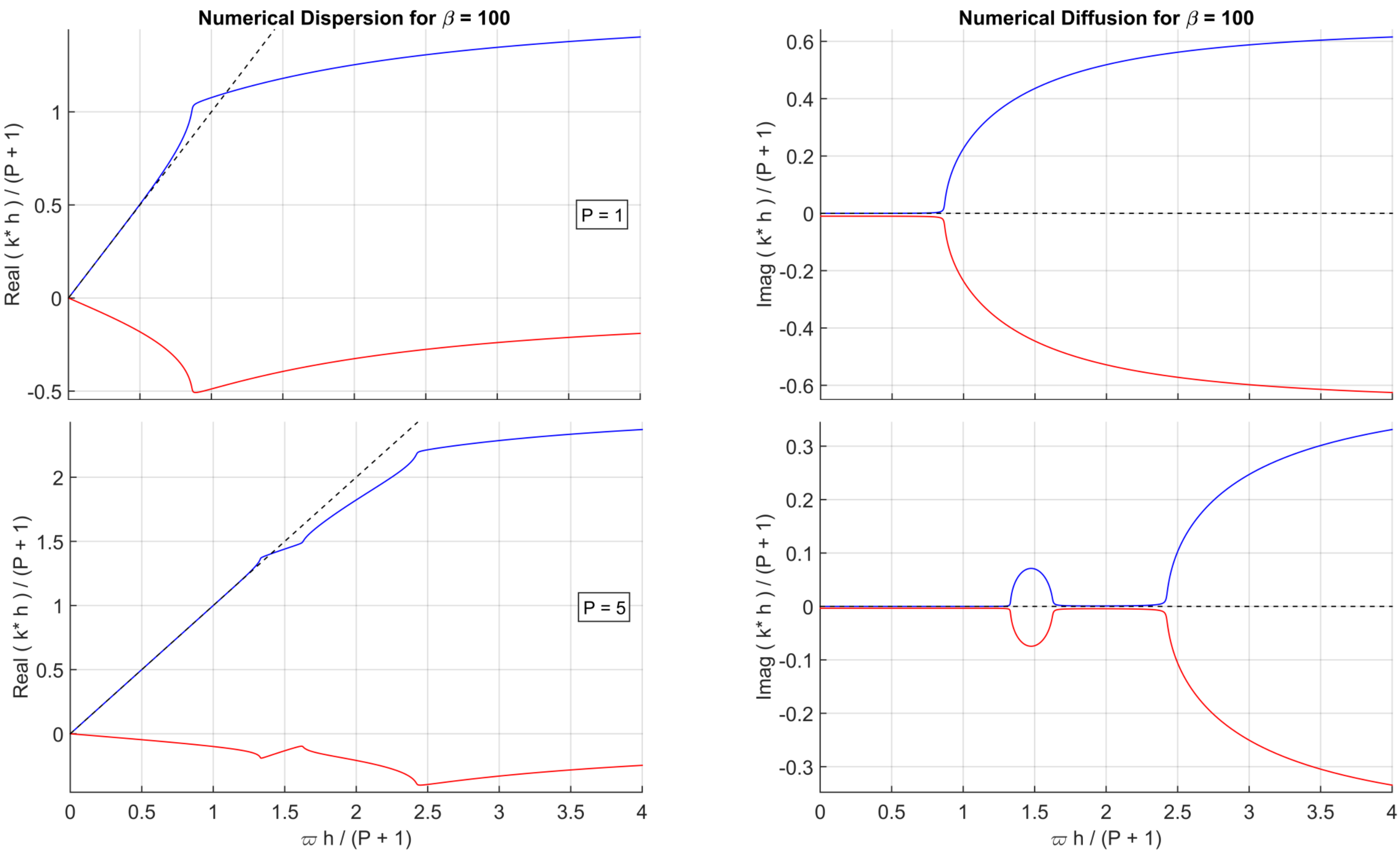

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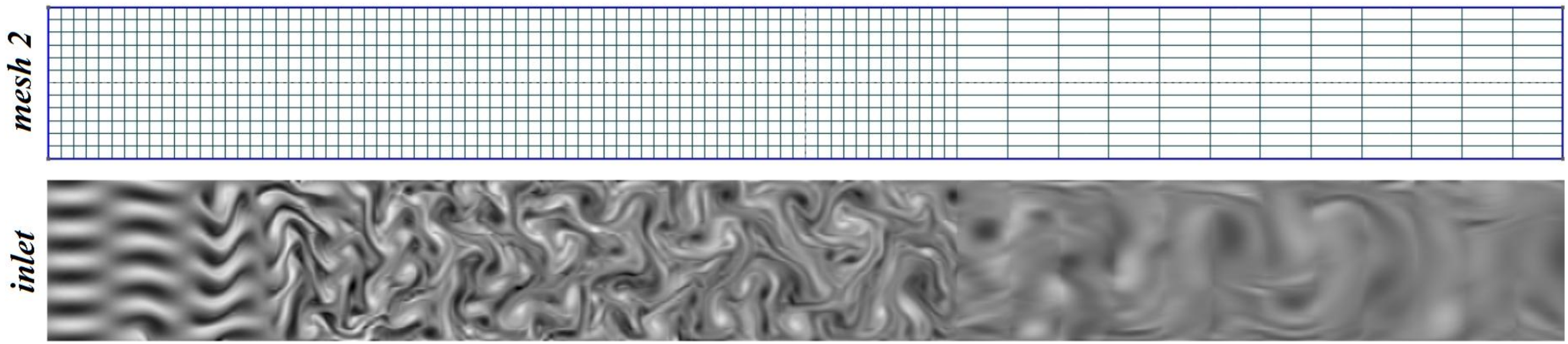
Numerical experiments in 2d grid turbulence



$$\rho = \rho_\infty \ ,$$
$$\rho u = \rho_\infty u_\infty \left[1 + A \sin(K y) \sin(\Omega t) \right] \ , \quad \rho v = 0 \ ,$$
$$E = p_\infty/(\gamma - 1) + \rho_\infty u_\infty^2$$

where $\rho_\infty = 1$, $u_\infty = 1$ are the free-stream density and mean flow velocity, while $p_\infty = \rho_\infty c_\infty^2/\gamma$ is the free-stream static pressure which is used to defined the flow's reference Mach number through the speed of sound $c_\infty = u_\infty \, \mathrm{Mach}^{-1}$. Moreover, the fluid's ratio of specific heats is set to $\gamma = 7/5$ and the parameters defining the inflow perturbations are given by $A = 1/2$, $K = 5$ and $\Omega = 1$.

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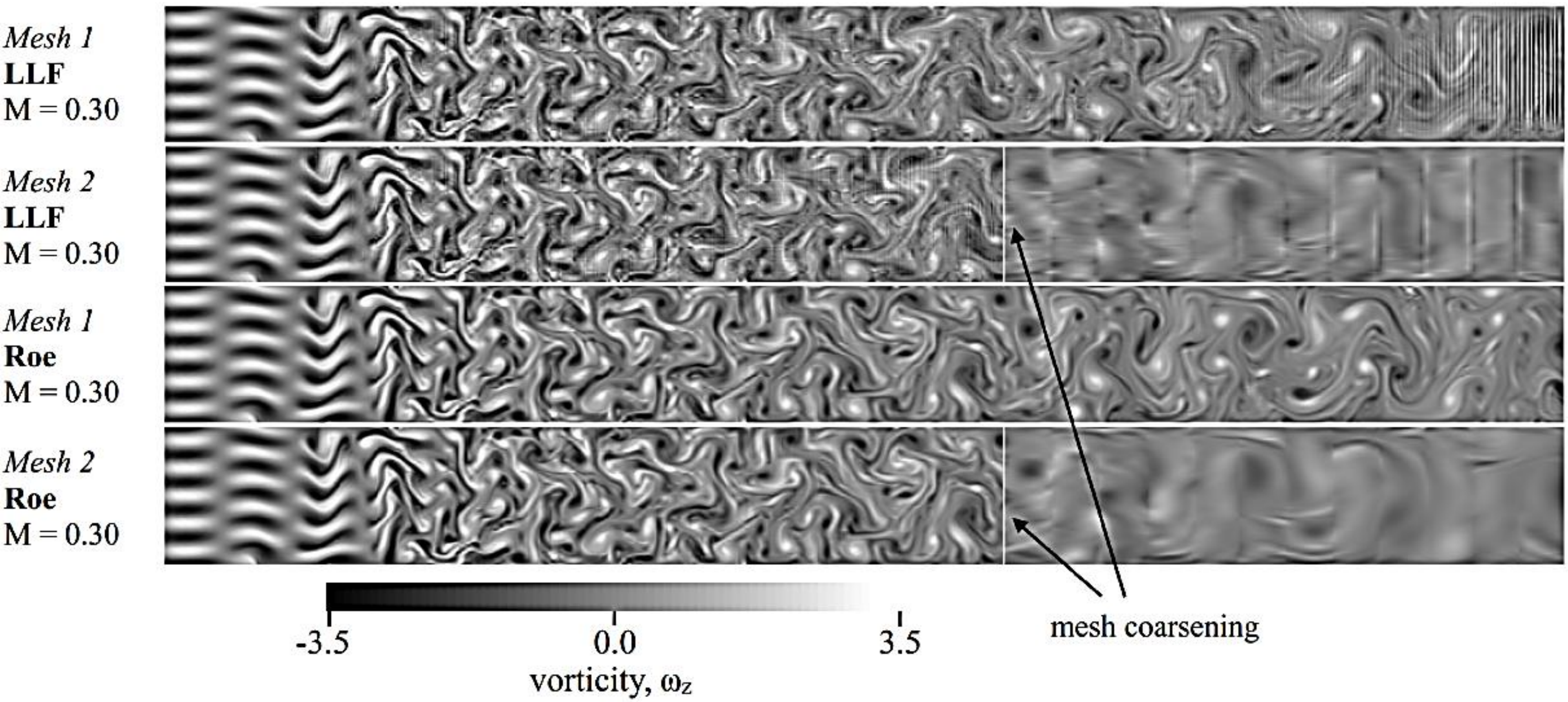

Mach 0.3

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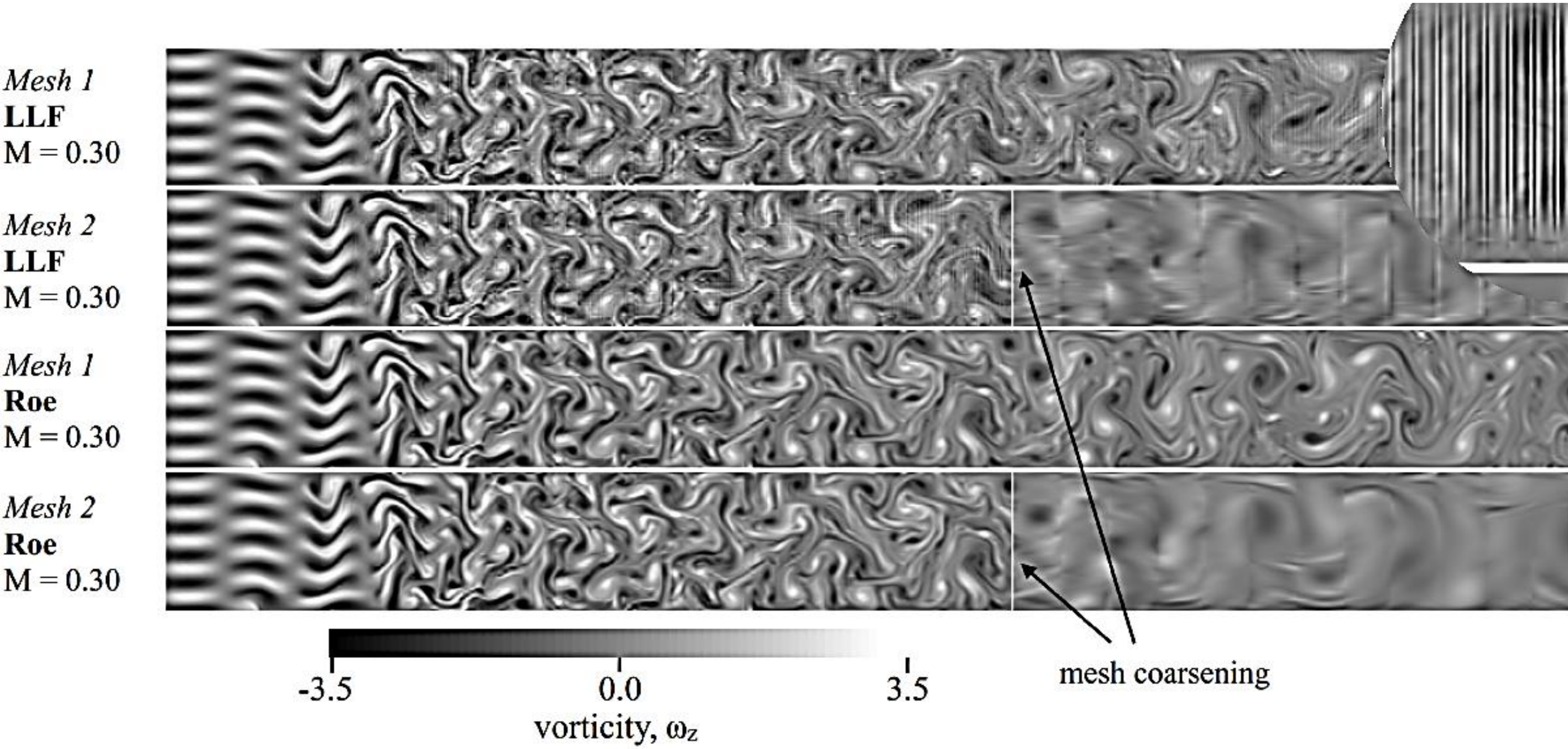

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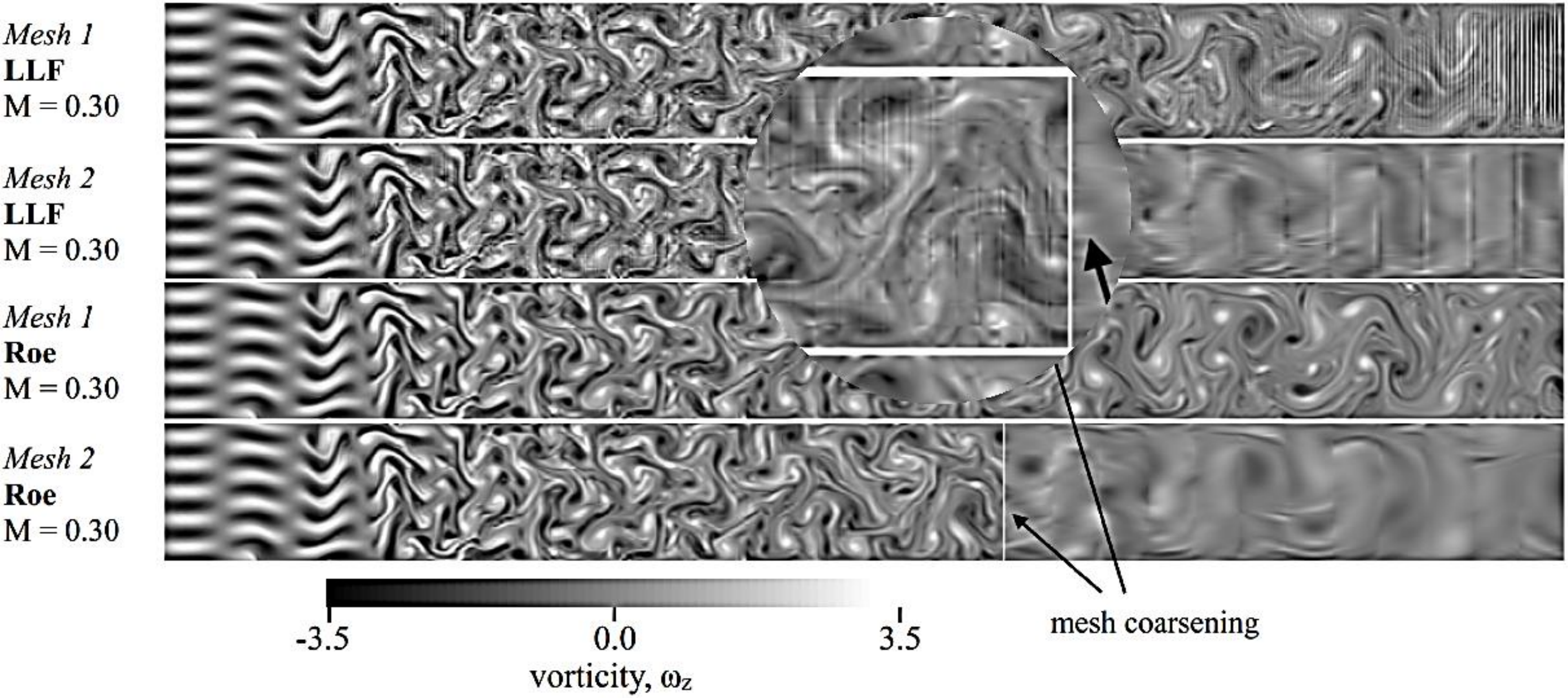

Mach 0.3

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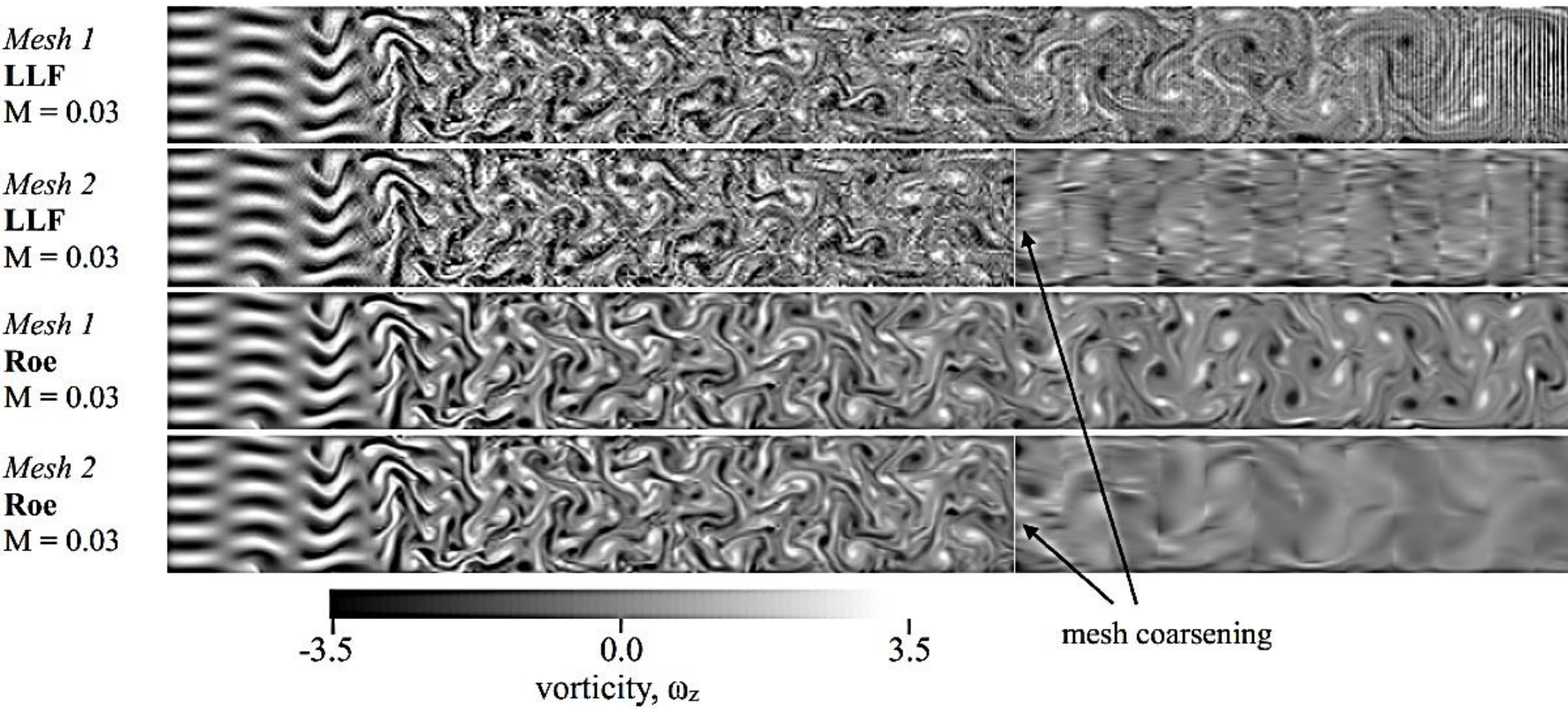

Mach 0.03

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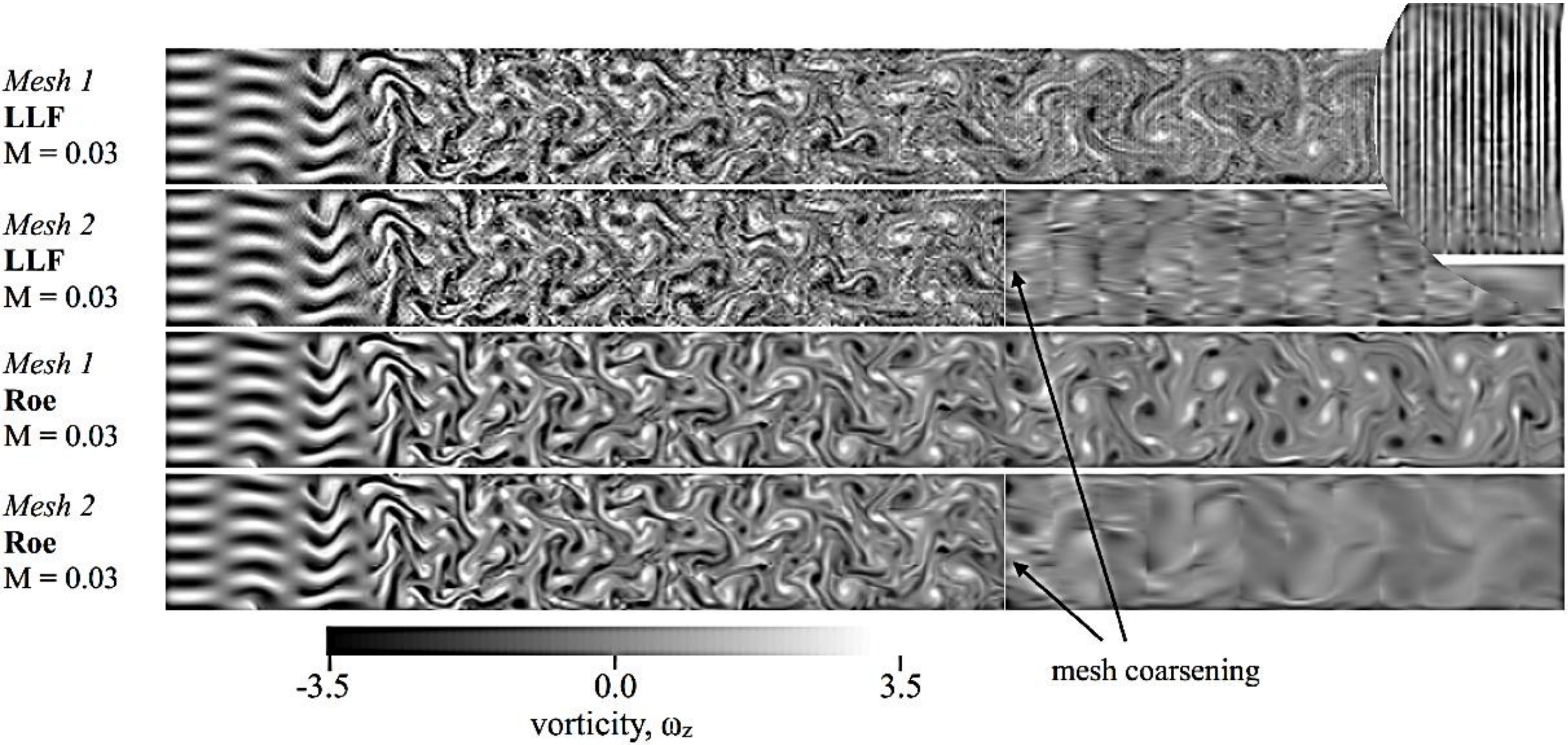

Mach 0.03

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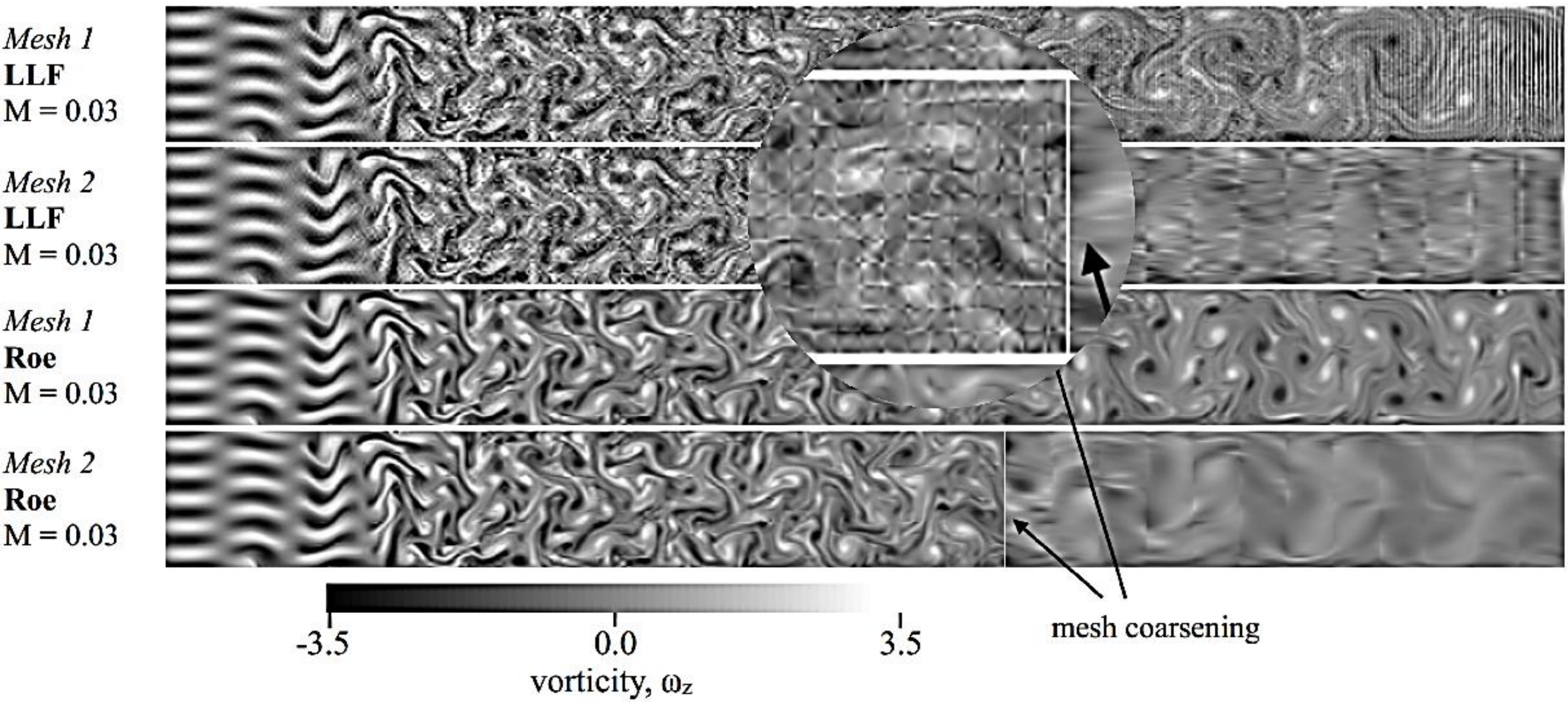

Mach 0.03

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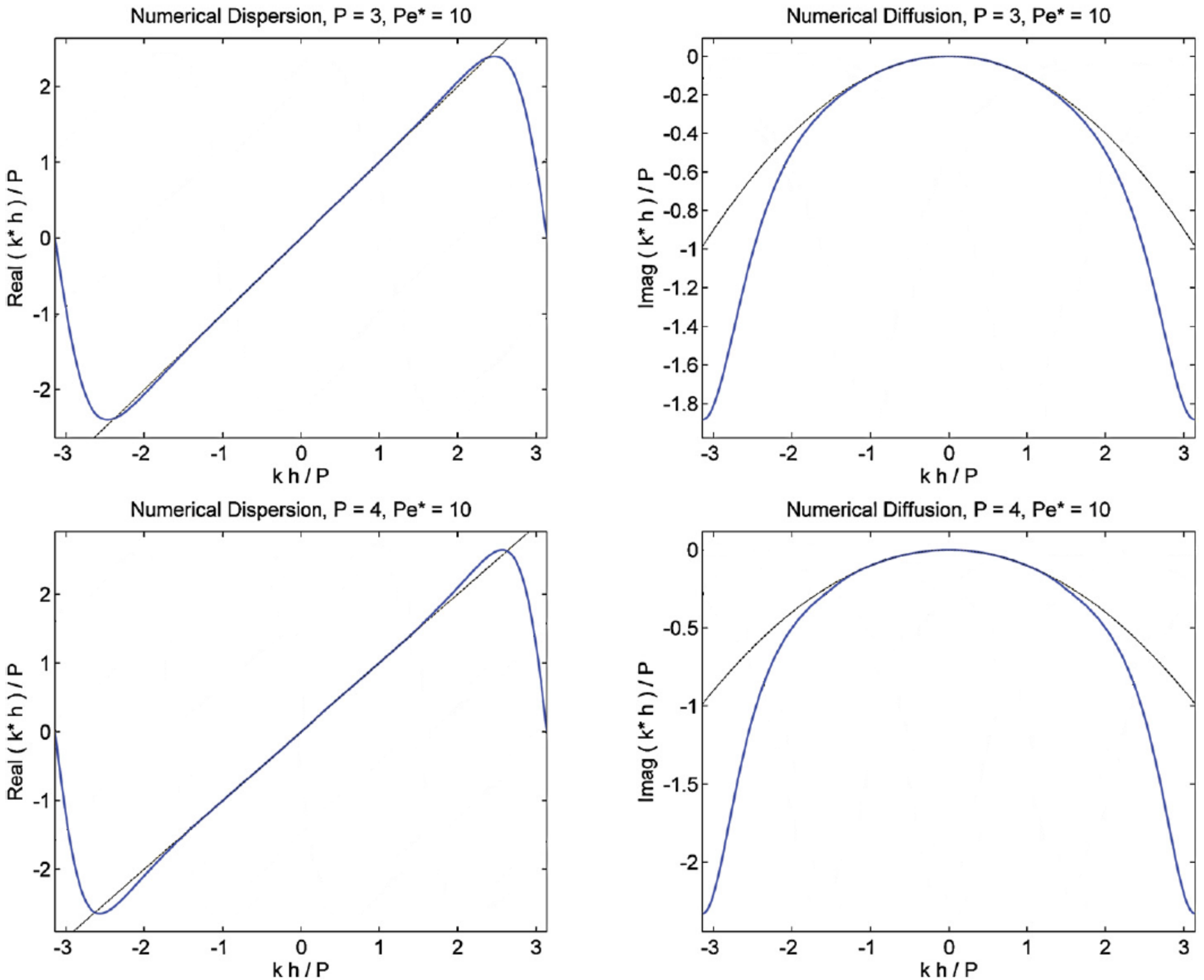
Temporal eigenanalysis of CG – 1D advection+diffusion



p = 3

$$\mathrm{Pe}^* = a\hbar/\mu = 10$$

p = 4

$$\mathrm{Pe}^* = a\hbar/\mu = 10$$

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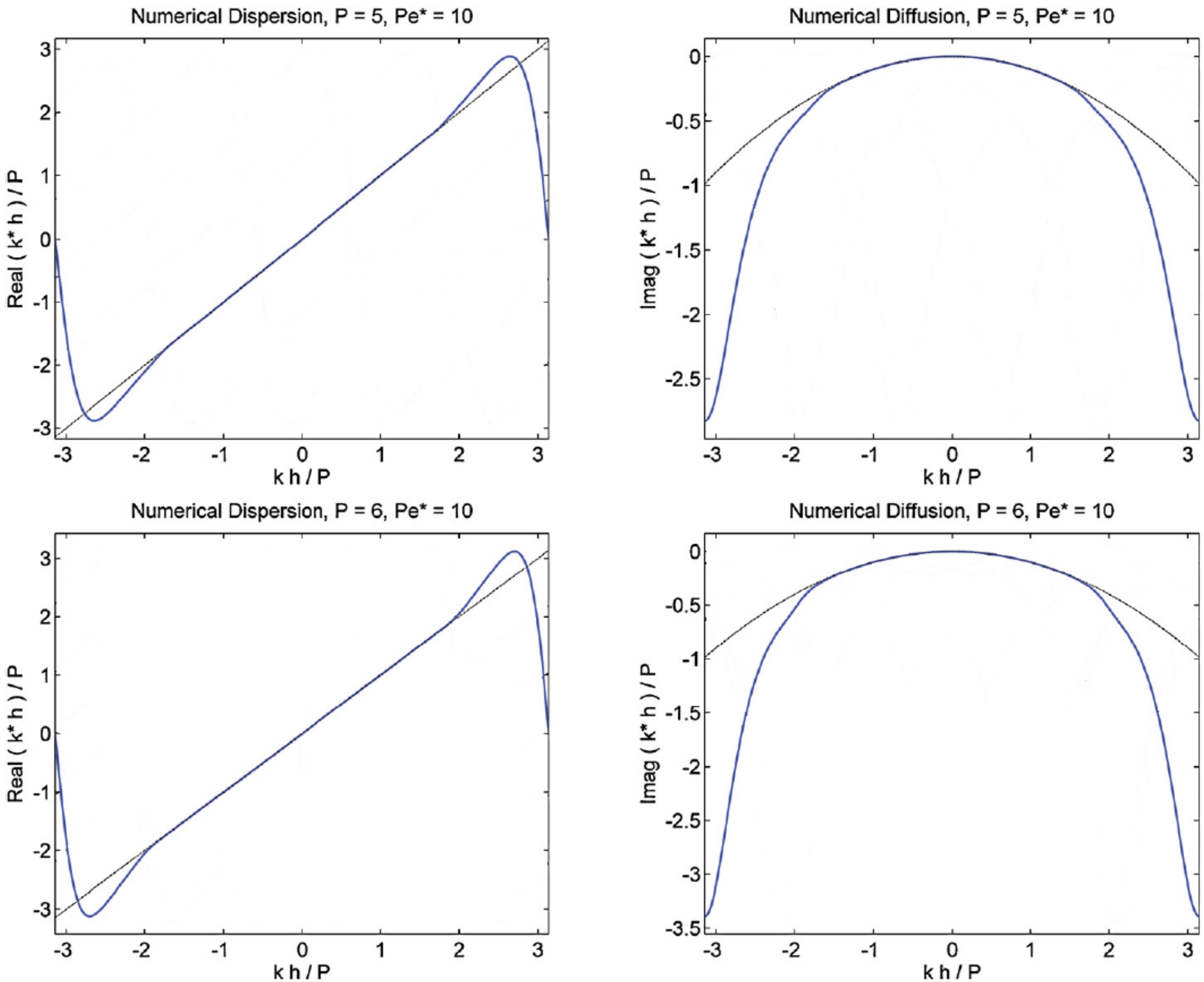


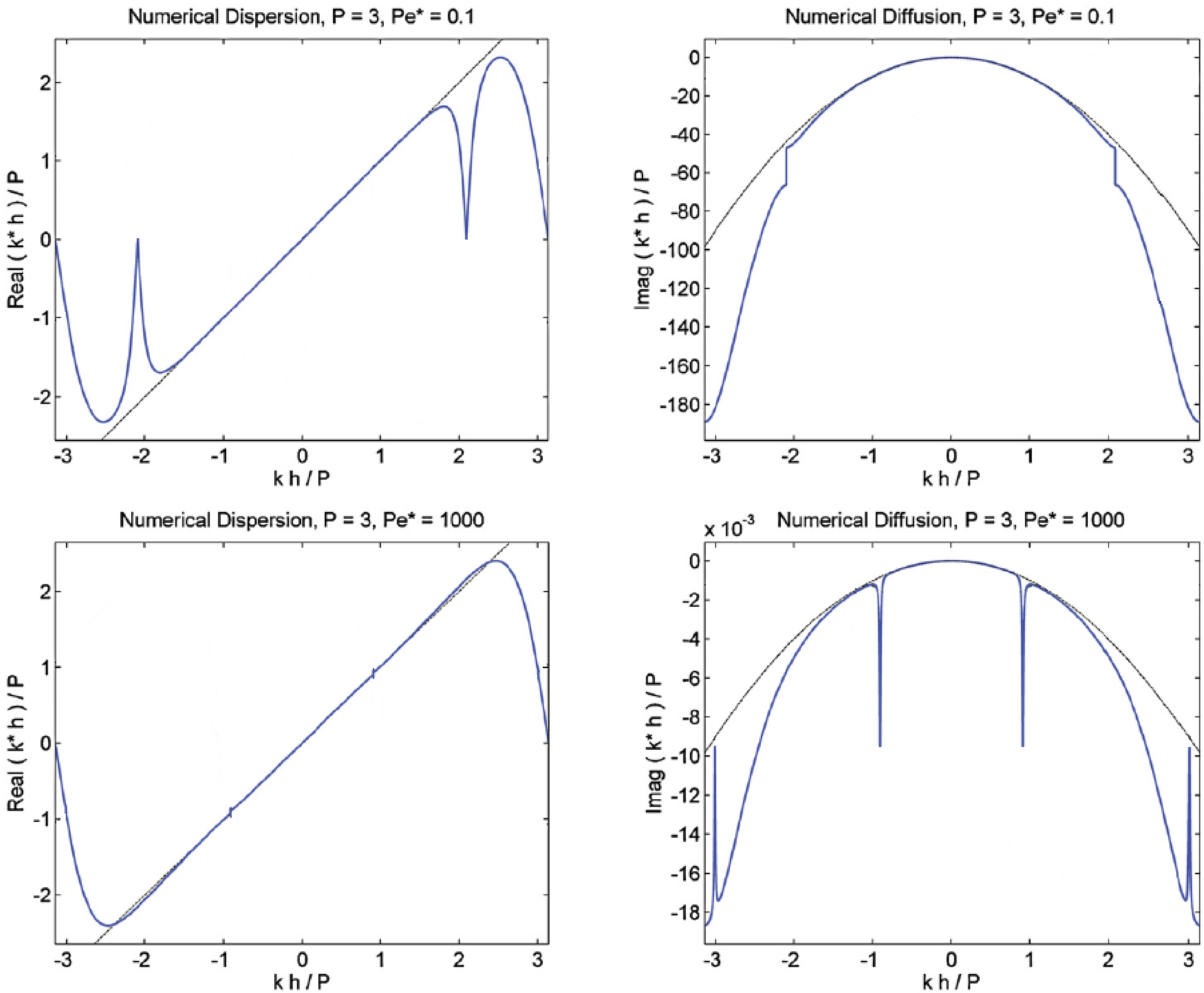
p = 5

$$\mathrm{Pe}^* = a\hbar/\mu = 10$$

p = 6

$$\mathrm{Pe}^* = a\hbar/\mu = 10$$

Temporal eigenanalysis of CG – 1D advection+diffusion



p = 3

$$\mathrm{Pe}^* = a\hbar/\mu = 0.1$$

p = 3

$$\mathrm{Pe}^* = a\hbar/\mu = 1000$$

Eigensolution analysis for CG – insights into SVV

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu \frac{\partial}{\partial x}\left(\mathcal{Q} \star \frac{\partial u}{\partial x}\right) \approx -\mu \sum_k k^2\, \hat{\mathcal{Q}}_k\, \hat{u}_k\, \exp(ikx)$$

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Eigensolution analysis for CG – insights into SVV

spectral vanishing viscosity

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strictly true for spectral methods

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REGULAR DIFFUSION RECOVERED WHEN $\mathcal{Q}_k = 1$ **for all** k

SVV KERNEL ENTRIES NORMALLY INCREASE FROM ZERO

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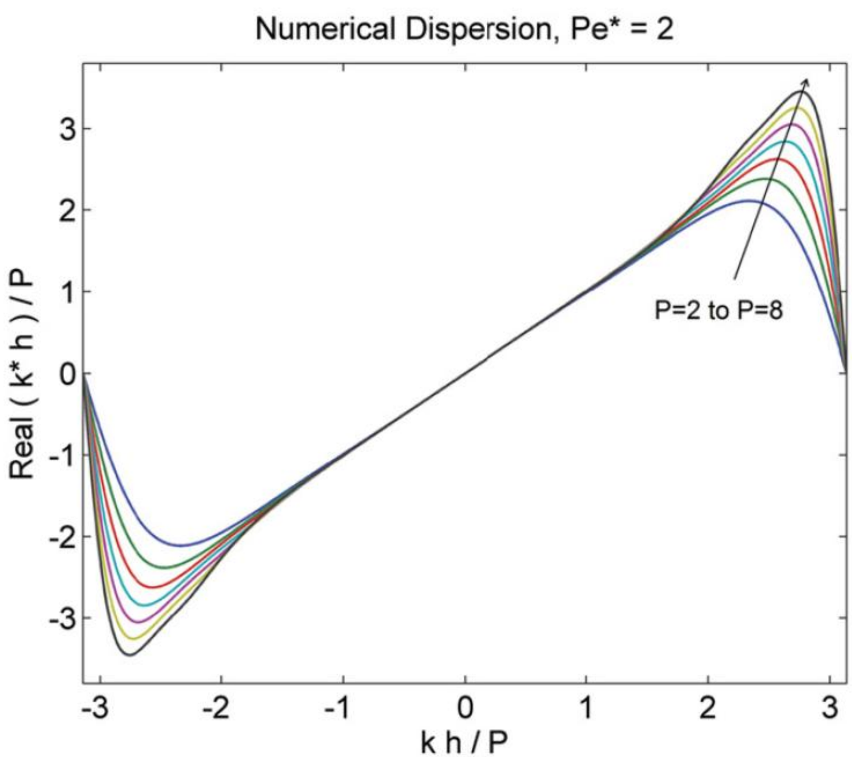

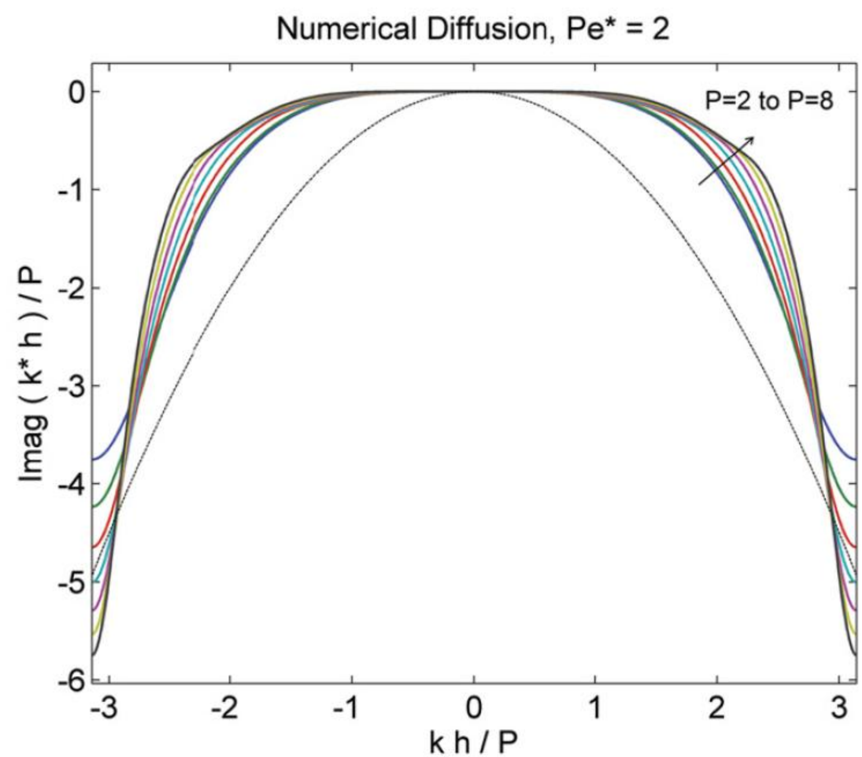

Proposed 'power kernel' : $\frac{\mathcal{Q}_k}{p} = \left(\frac{k}{p}\right)^{p_{svv}}$, $p_{svv} = p/2$

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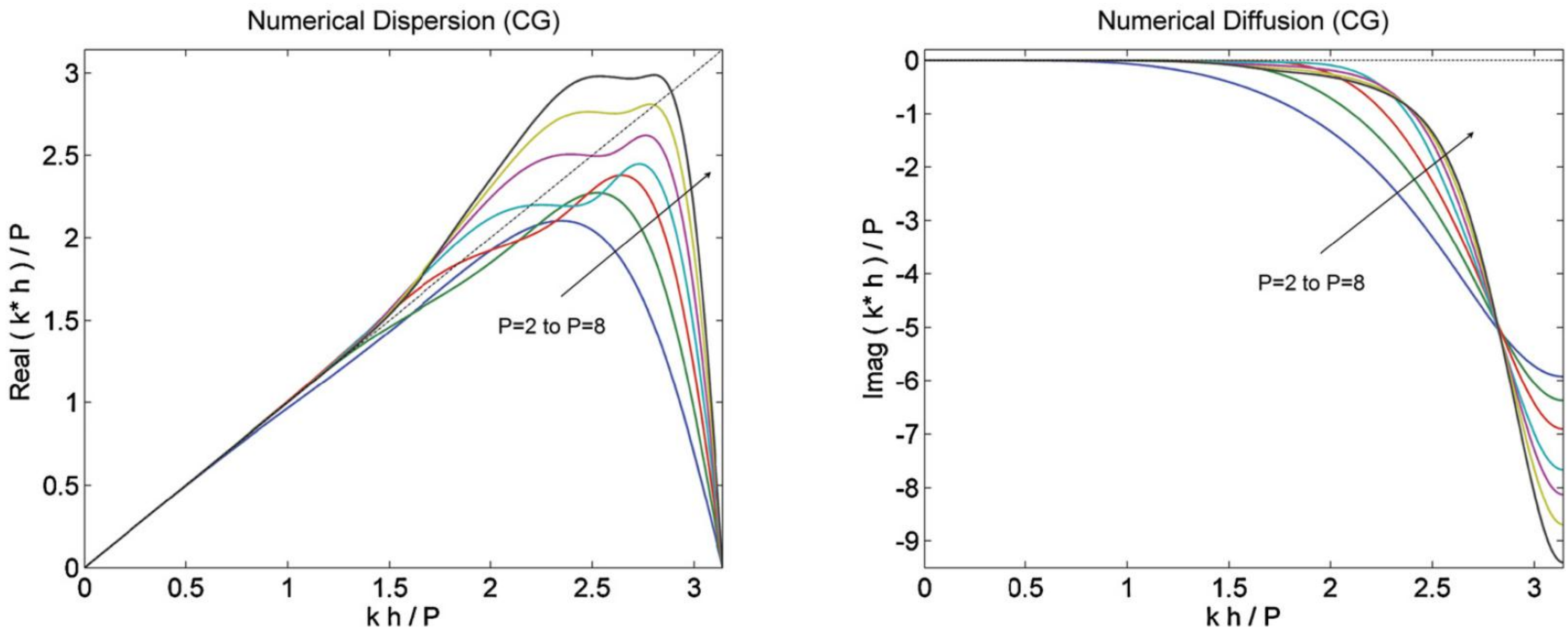
Temporal eigenanalysis for CG – a Péclet-free SVV

$$\text{Using} \quad \mu \propto \frac{ah}{p} \implies \text{fixed } \mathrm{Pe}^* = a\hbar/\mu$$

(optimized SVV kernel to mimic DG)



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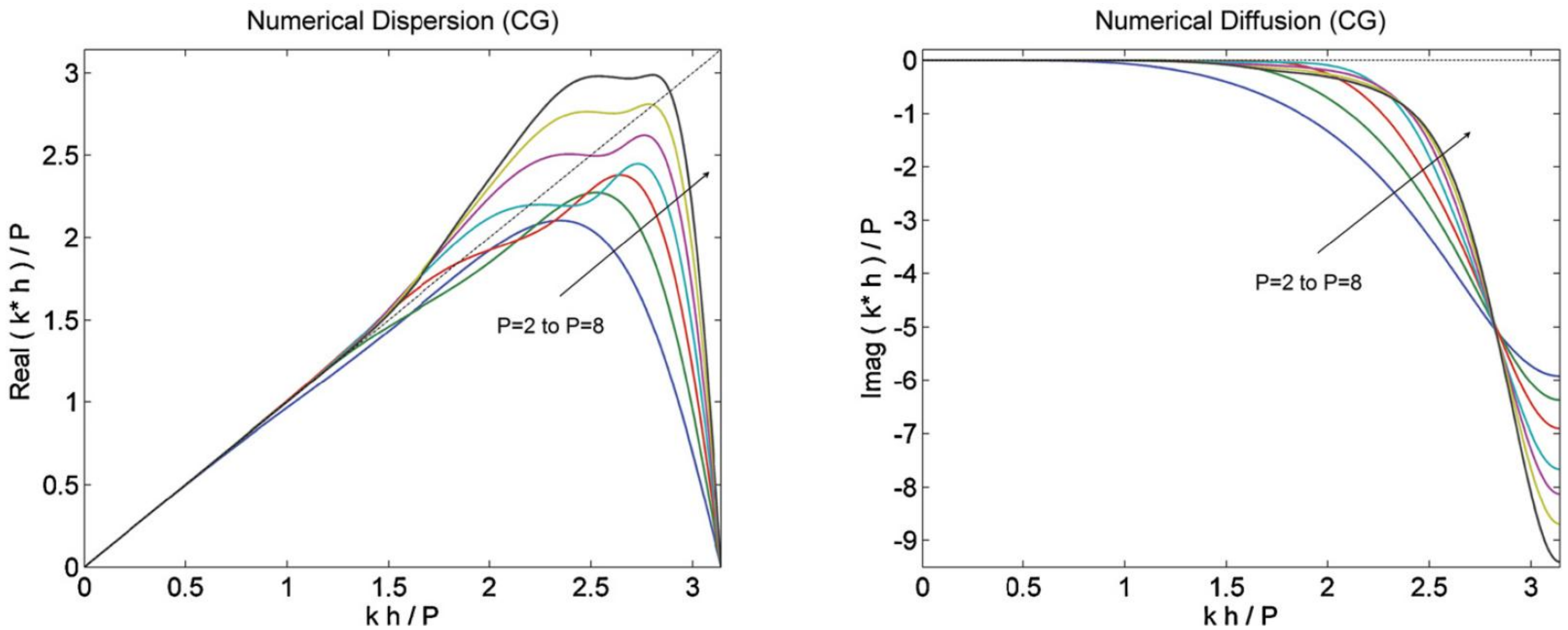
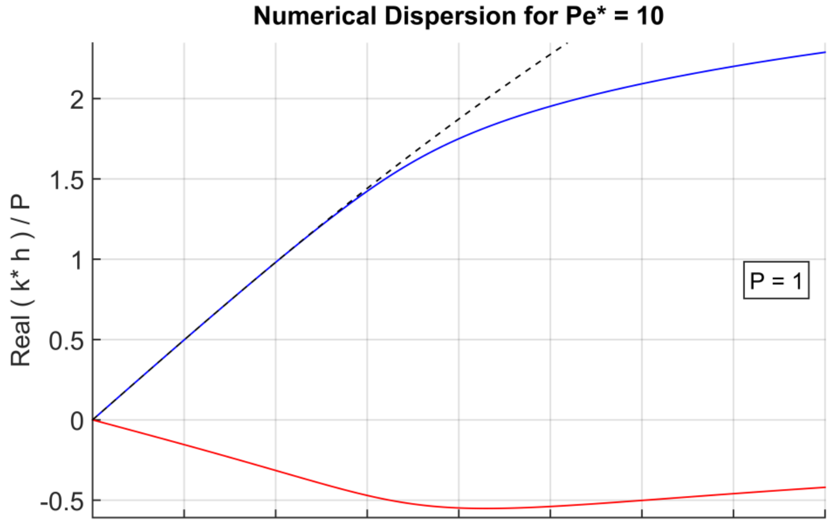
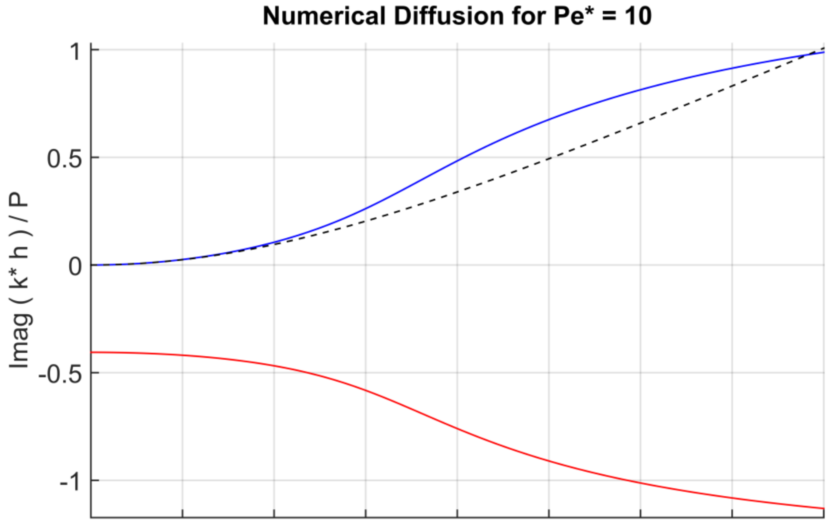


Spatial eigenanalysis for CG – 1D advection+diffusion





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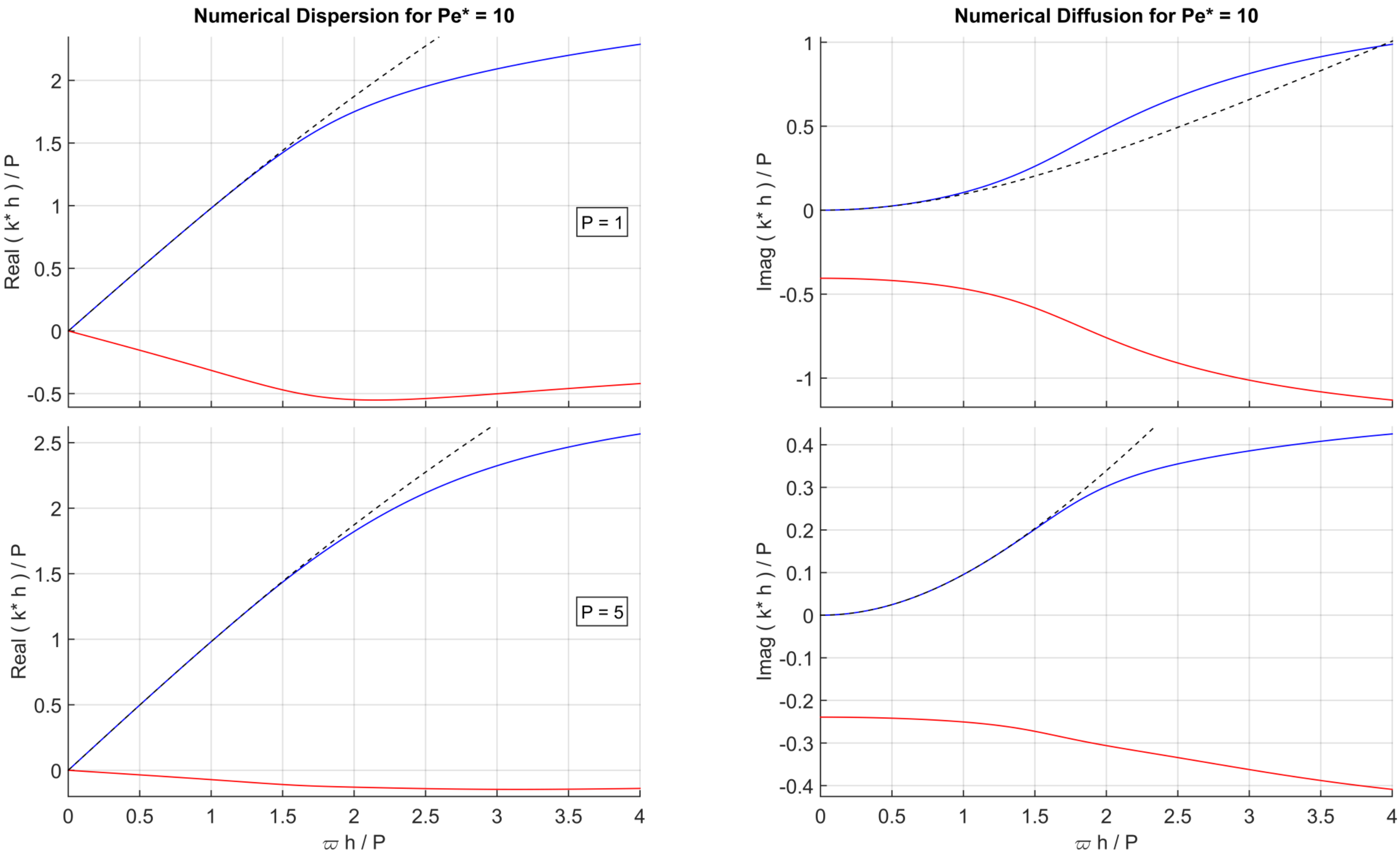

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Spatial eigenanalysis for CG – pure advection limit

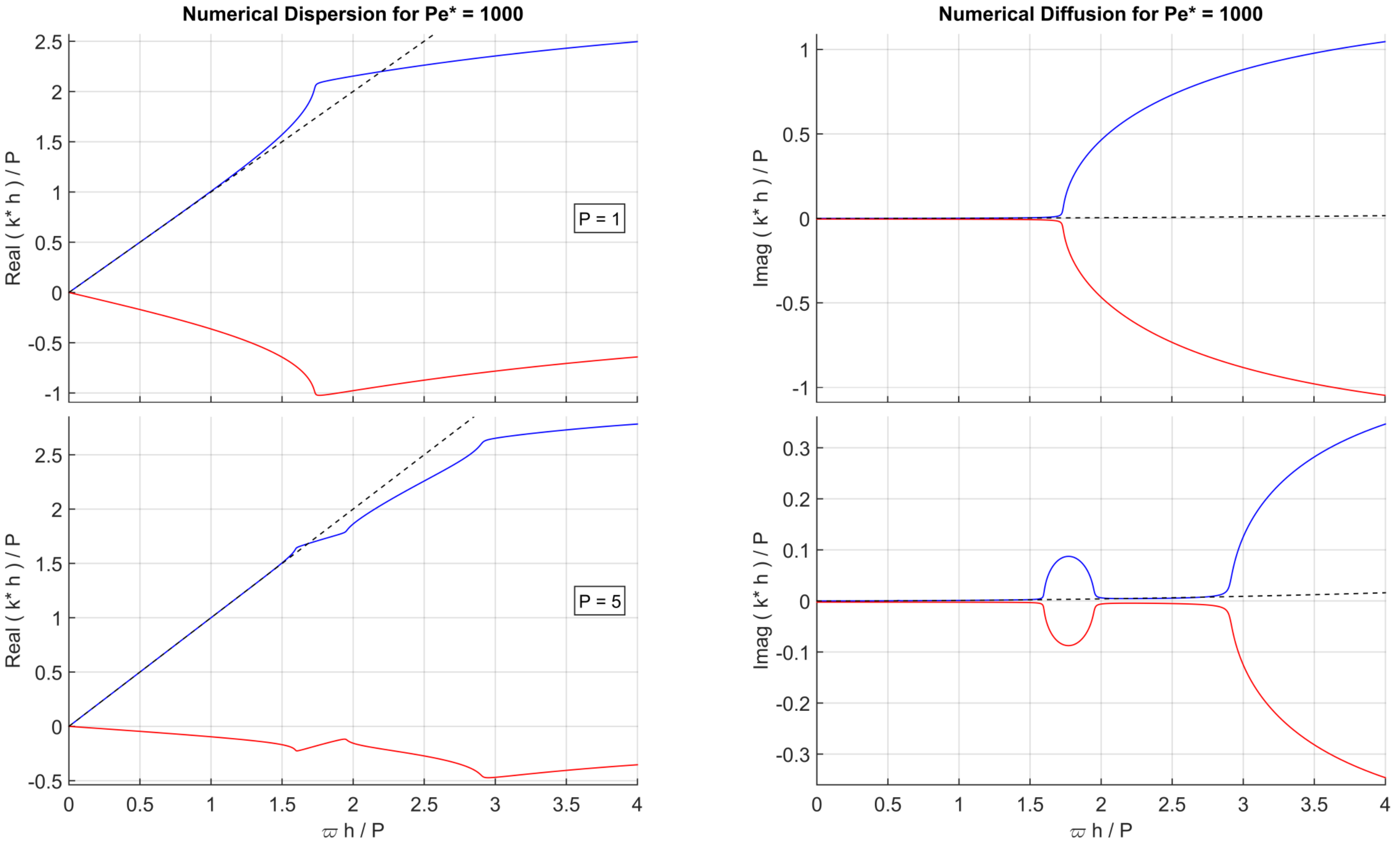
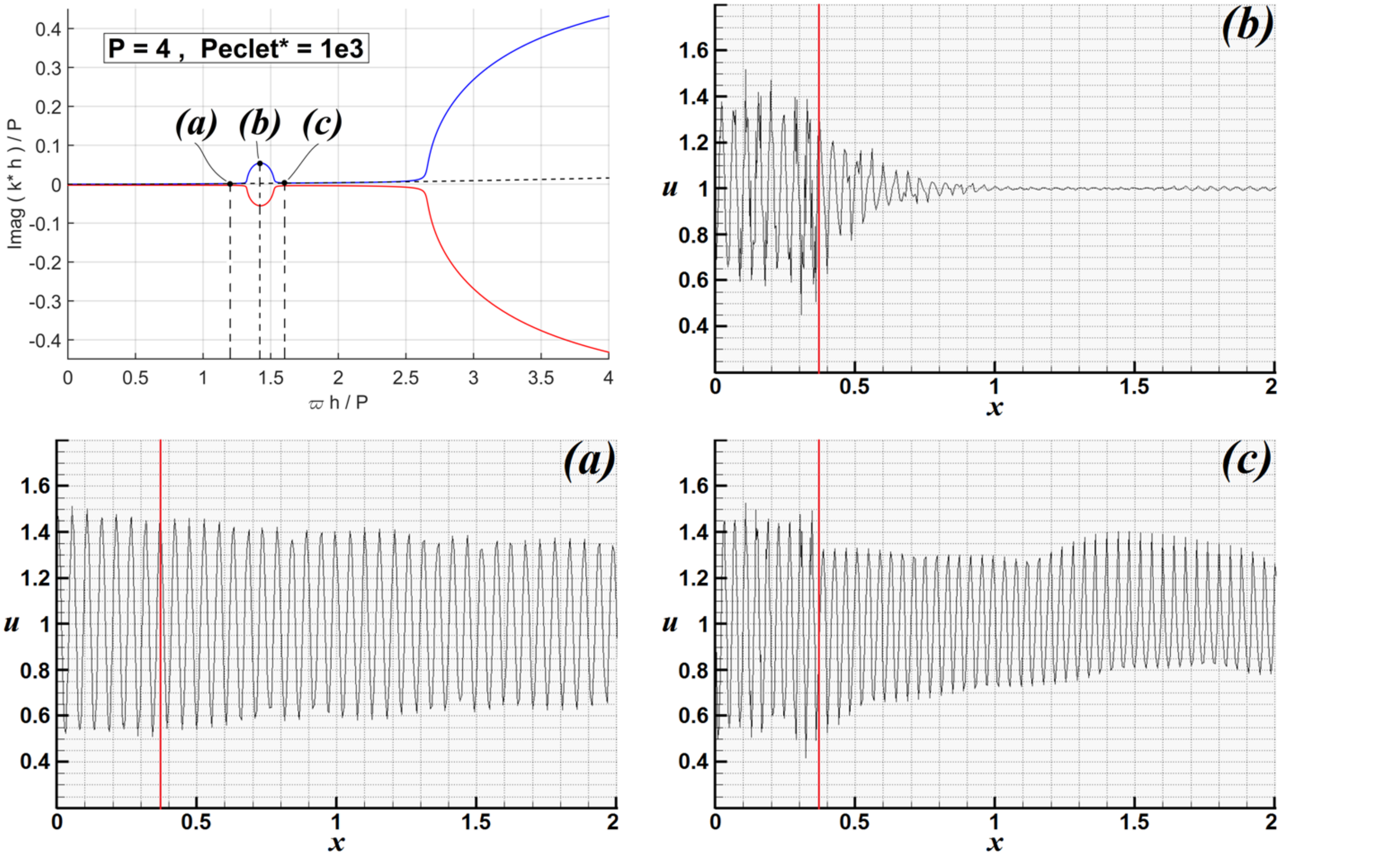


Illustration of dissipation bubble effects



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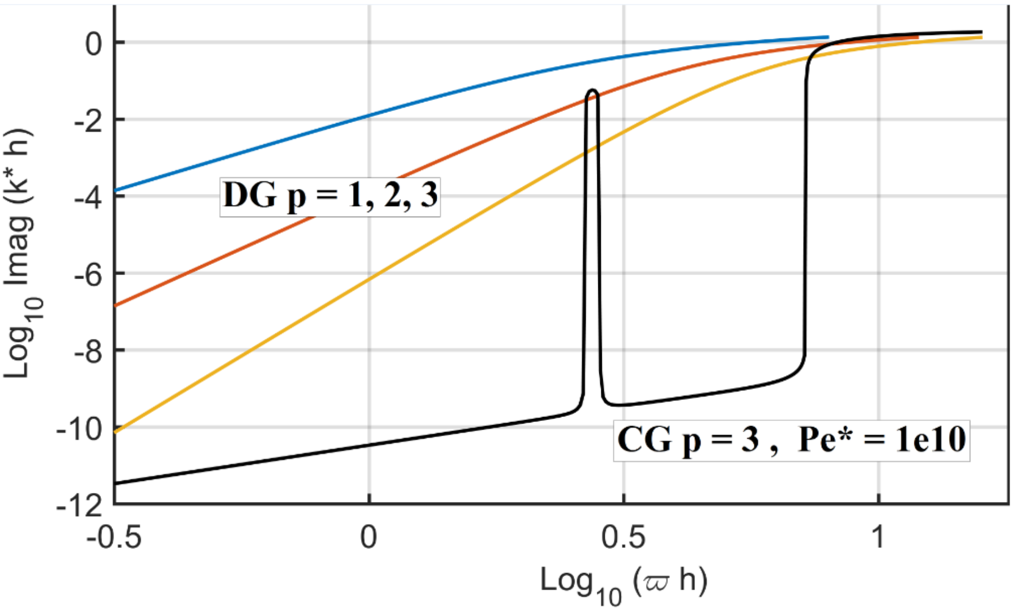
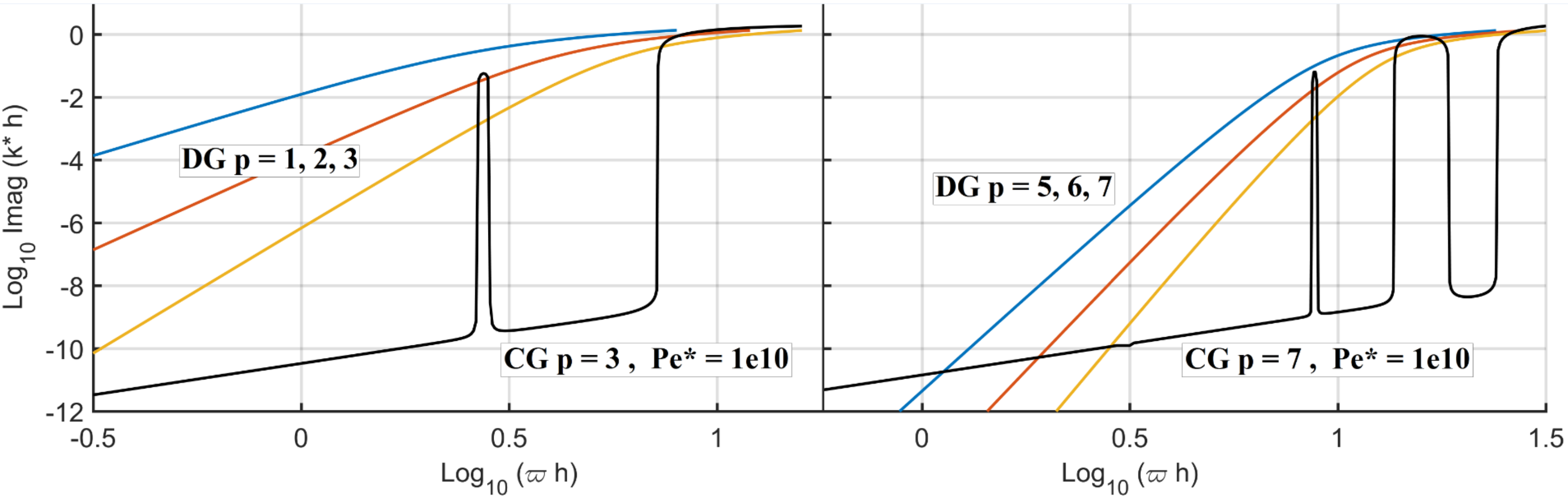
Spatial eigenanalysis of CG – 1D advection+SVV



DG & CG dissipation curves in the limit of pure advection (log-log plots)

Spatial eigenanalysis of CG – 1D advection+SVV



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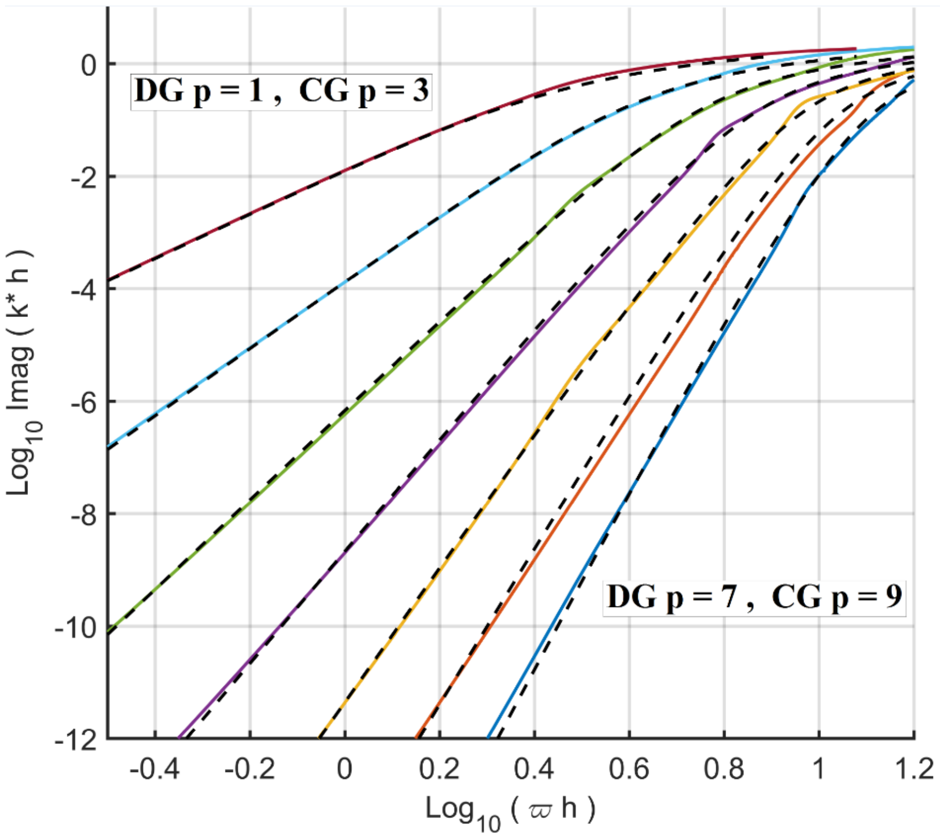
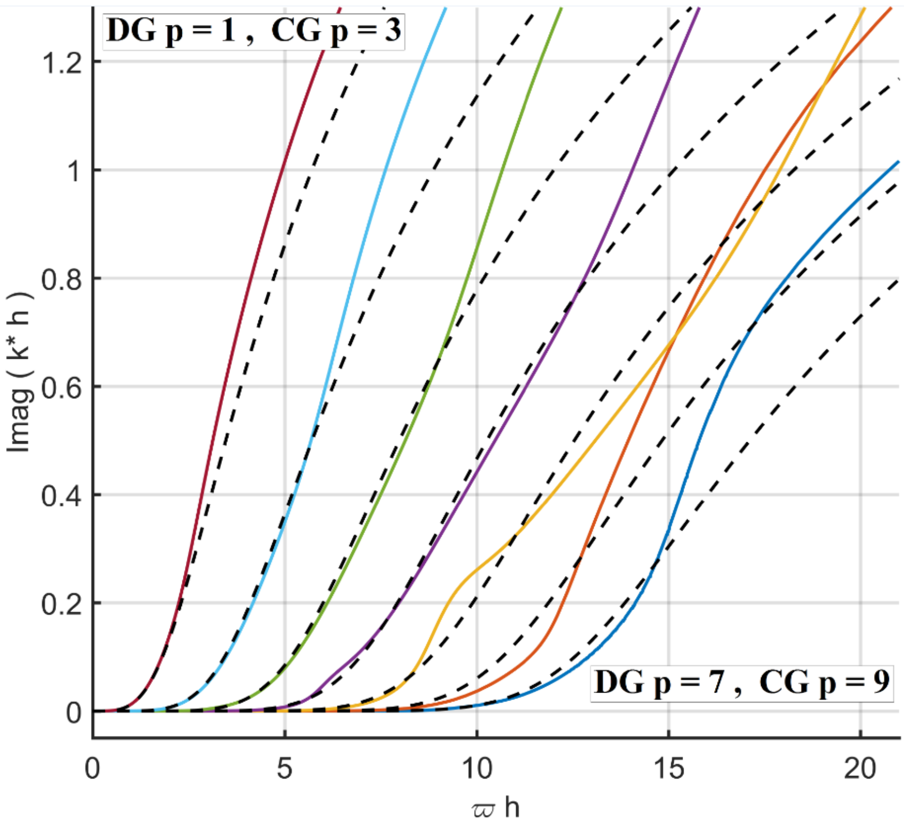
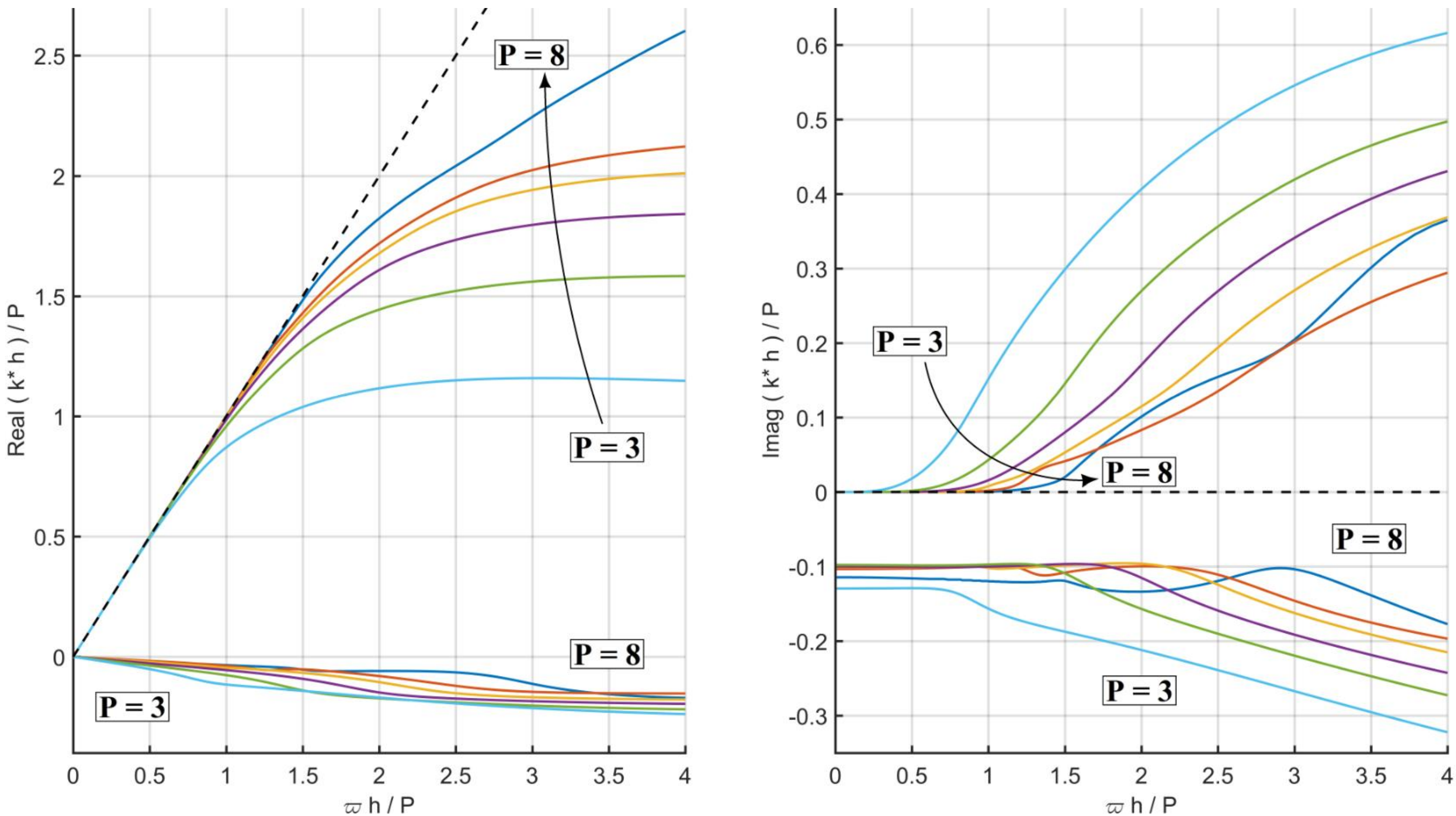


(log-log)



(lin-lin)

Spatial eigenanalysis of CG – 1D advection+SVV



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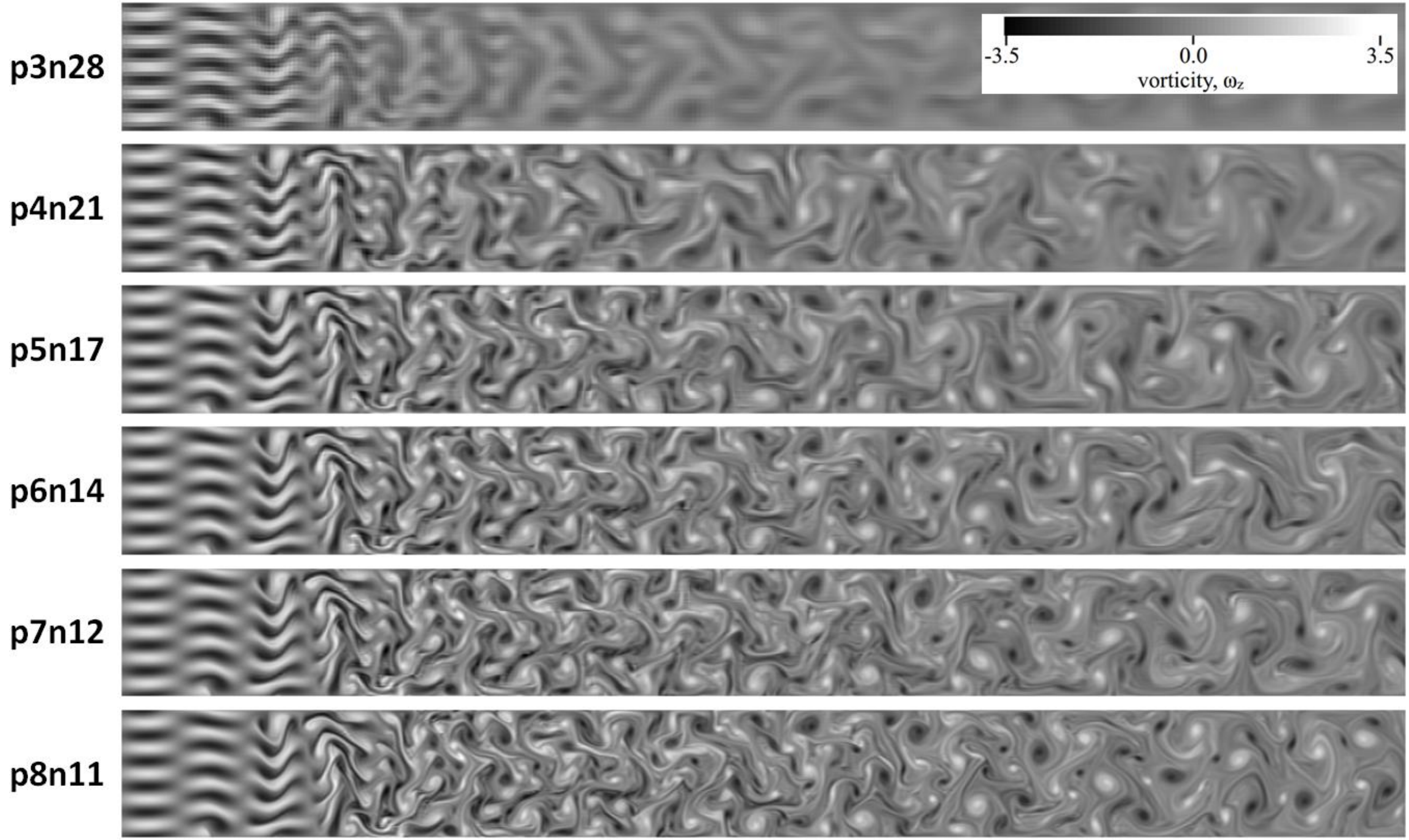
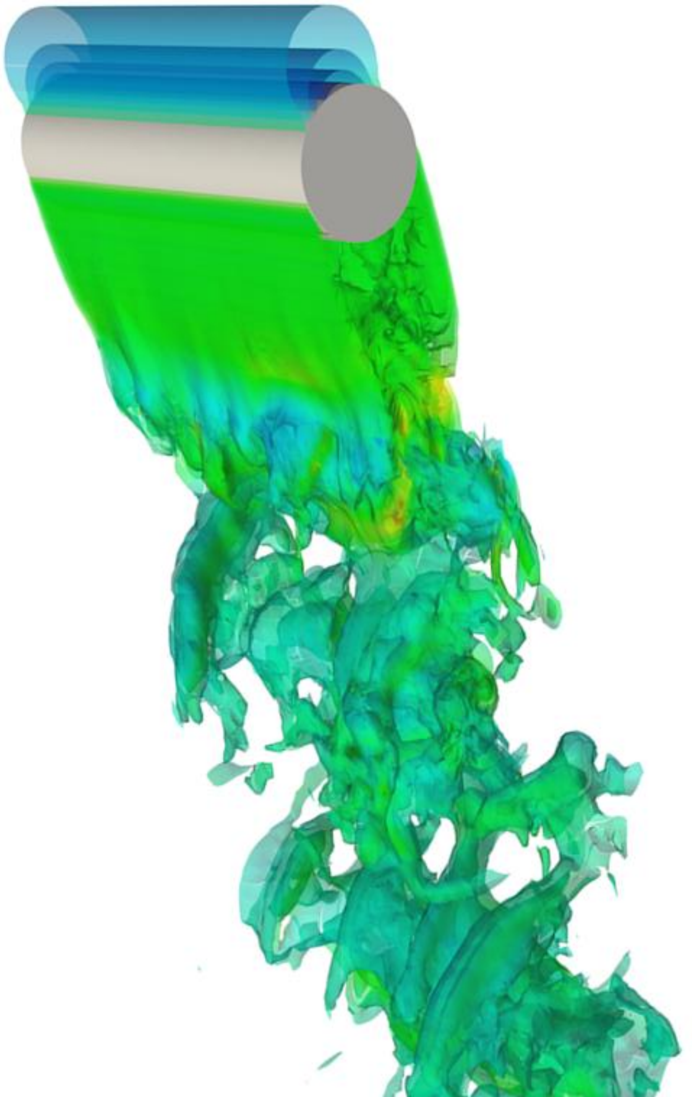
Numerical experiments in 2d grid turbulence via CG+SVV



Conclusions & outlook

- Temporal vs Spatial eigenanalysis
- Beware of spurious reflections and dissipation bubbles!
- Pick complete Riemann solvers for DG
- Use robust SVV operators with CG
- Avoid abrupt coarsening (in 2d, 3d)
- Favour moderately high polynomial orders along with coarser grids and use polynomial dealiasing!



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Questions