The Method of Moving Frames for Maxwell's Equations on Curved Surfaces
with Nektar++

Ehsan Kazemi
Sehun Chun

Overview

- Part 1: *Formulation of DG scheme based on MMF for Maxwell's Equations*
- Part 2: *Numerical Results: Applications to Invisible Cloak and ELF*

Maxwell's Equations

Consider the time-dependent Maxwell's equations on curved surfaces without source terms such as

$$\hat{\varepsilon}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \qquad \hat{\mu}\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E},$$

- $\mathbf{E}$ is the electric field
- $\mathbf{H}$ is the h-field
- $\hat{\varepsilon} = \begin{bmatrix} \varepsilon^{xx} & \varepsilon^{xy} \\ \varepsilon^{yx} & \varepsilon^{yy} \end{bmatrix}$ permittivity tensor
- $\hat{\mu} = \begin{bmatrix} \mu^{xx} & \mu^{xy} \\ \mu^{yx} & \mu^{yy} \end{bmatrix}$ permeability tensor

Maxwell's Equations

Consider the time-dependent Maxwell's equations on curved surfaces without source terms such as

$$\hat{\varepsilon}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \qquad \hat{\mu}\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E},$$

- $\mathbf{E}$ is the electric field
- $\mathbf{H}$ is the h-field
- $\hat{\varepsilon} = \begin{bmatrix} \varepsilon^{xx} & \varepsilon^{xy} \\ \varepsilon^{yx} & \varepsilon^{yy} \end{bmatrix}$ permittivity tensor
- $\hat{\mu} = \begin{bmatrix} \mu^{xx} & \mu^{xy} \\ \mu^{yx} & \mu^{yy} \end{bmatrix}$ permeability tensor

If the off-diagonal terms of these tensors are nonzero, the equations are off-diagonally anisotropic.
For diagonal tensors, with non-equal diagonal terms, the equations are axially anisotropic.

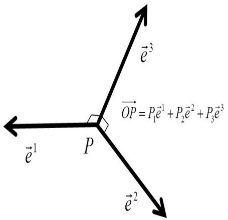
DG Scheme for Maxwell's Equations

- The DG method has been acclaimed for its flexibility and accuracy, even for complex geometry with complex boundaries.
- The stable and accurate performance of the DG largely owes to the use of the upwind flux between the interfaces of elements that satisfies the Rankine-Hugoniot condition for conservational laws.
- The universal upwind flux for general anisotropic media has been unknown, because of the difficulty of satisfying the Rankine-Hugoniot conditions in anisotropic media.

DG Scheme for Maxwell's Equations

- It has been demanded for the studies of important modern electromagnetic phenomena for example, metamaterial physical phenomena such as invisible cloak and electromagnetic wave propagation in the earth-ionosphere waveguide.
- The use the central flux, which will require a reduction in accuracy and stability requirements.
- There are some methods for the derivation of upwind flux in anisotropic media but the extention to higher dimensions is not clear.

Moving Frames



Construct moving frames consisting of three vectors at a point P : $(\mathbf{e}^1)$ and second moving frame $(\mathbf{e}^2)$ lie on the x-y plane and in arbitrary direction, but they are orthogonal.
The third $(\mathbf{e}^3)$ is aligned along the z-direction, orthogonal to $\mathbf{e}^1$ and $\mathbf{e}^2$. The location of a point P, as $\mathbf{P} = P_x\mathbf{x} + P_y\mathbf{y} + P_z\mathbf{z}$ is transformed in the moving frames $(\mathbf{e}^i)$ such as

$$\mathbf{P} = P_1\mathbf{e}^1 + P_2\mathbf{e}^2 + P_3\mathbf{e}^3.$$

Converting from anisotropy to isotropy

We start from Cartesian coordinate system as follows:

$$\hat{\varepsilon}\mathbf{E}=[\mathbf{x}\ \mathbf{y}]\begin{bmatrix} \varepsilon^{xx} & \varepsilon^{xy} \\ \varepsilon^{yx} & \varepsilon^{yy} \end{bmatrix}\begin{bmatrix} E^x \\ E^y \end{bmatrix}=(\varepsilon^{xx}E^x+\varepsilon^{xy}E^y)\mathbf{x}+(\varepsilon^{yx}E^x+\varepsilon^{yy}E^y)\mathbf{y}.$$

Let $\mathbf{e}^1$ be aligned along this anisotropy and $\mathbf{e}^2$ is orthogonal to $\mathbf{e}^1$. In this set of moving frames we obtain

$$\hat{\varepsilon}\mathbf{E} = \left[\mathbf{e}^1\ \mathbf{e}^2\right]\begin{bmatrix} \varepsilon^1 & 0 \\ 0 & \varepsilon^2 \end{bmatrix}\begin{bmatrix} E^1 \\ E^2 \end{bmatrix} = \varepsilon^1 E^1 \mathbf{e}^1 + \varepsilon^2 E^2 \mathbf{e}^2,$$

where ε^1 and ε^2 are the eigenvalues of the permittivity tensor $\hat{\varepsilon}$.

Converting from anisotropy to isotropy

We start from Cartesian coordinate system as follows:

$$\hat{\varepsilon}\mathbf{E}=[\mathbf{x}\ \mathbf{y}]\left[\begin{array}{cc}\varepsilon^{xx} & \varepsilon^{xy}\\ \varepsilon^{yx} & \varepsilon^{yy}\end{array}\right]\left[\begin{array}{c}E^{x}\\ E^{y}\end{array}\right]=(\varepsilon^{xx}E^{x}+\varepsilon^{xy}E^{y})\mathbf{x}+(\varepsilon^{yx}E^{x}+\varepsilon^{yy}E^{y})\mathbf{y}.$$

Let $\mathbf{e}^1$ be aligned along this anisotropy and $\mathbf{e}^2$ is orthogonal to $\mathbf{e}^1$. In this set of moving frames we obtain

$$\hat{\varepsilon}\mathbf{E}=\left[\mathbf{e}^1\ \mathbf{e}^2\right]\left[\begin{array}{cc}\varepsilon^{1} & 0\\ 0 & \varepsilon^{2}\end{array}\right]\left[\begin{array}{c}E^{1}\\ E^{2}\end{array}\right]=\varepsilon^1E^1\mathbf{e}^1+\varepsilon^2E^2\mathbf{e}^2,$$

where ε^1 and ε^2 are the eigenvalues of the permittivity tensor $\hat{\varepsilon}$. Furthermore, by using the new moving frames $\mathbf{d}^1=(\sqrt{\varepsilon^1}/\sqrt{\varepsilon^2})\mathbf{e}^1$ and $\mathbf{d}^2=(\sqrt{\varepsilon^2}/\sqrt{\varepsilon^1})\mathbf{e}^2$, the isotropic representation of $\hat{\varepsilon}\mathbf{E}$ is obtained as follows:

$$\hat{\varepsilon}\mathbf{E}=\sqrt{\varepsilon^1\varepsilon^2}\left(E^1\mathbf{d}^1+E^2\mathbf{d}^2\right).$$

Converting from anisotropy to isotropy

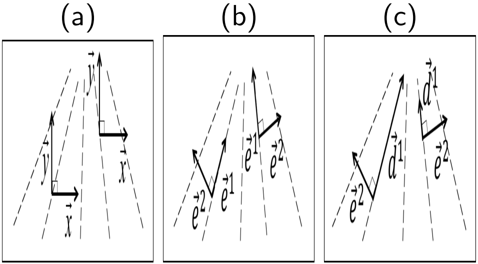


Figure: (a) Off-diagonal anisotropic expression in the Cartesian coordinate system, (b) axial anisotropic representation and (c) isotropic representation in a local reference frame (right). The dashed lines represent anisotropy.

Moving frames for real and imaginary tensors

In general, in the perspective of energy dissipation, the permittivity tensor is expressed as a complex tensor as $\hat{\varepsilon} = \hat{\varepsilon}_r + \hat{\sigma}/i\omega$, where $\hat{\varepsilon}_r$ is the real permittivity tensor and $\hat{\sigma}$ is the electric conductivity tensor for frequency ω. Thus, the moving frames that should be aligned along an anisotropy should also be expressed in a complex form as follows: for $1 \leq i \leq 3$,

$$\mathbf{e}^i = \mathbf{e}^i_\varepsilon + i\mathbf{e}^i_\sigma,$$

where $\mathbf{e}^i_\varepsilon$ and $\mathbf{e}^i_\sigma$ are aligned along the anisotropic direction of real permittivity and the electric conductivity, respectively.

Maxwell's equations in moving frames coordinates

Expand $\mathbf{E}$ and $\mathbf{H}$ in TM mode such that $\mathbf{E} = E^3\mathbf{e}^3$ and $\mathbf{H} = H^1\mathbf{e}^1 + H^2\mathbf{e}^2$. By substitution, the Maxwell's equations yield, for $i = 1,\ 2$,

$$\mu^i \frac{\partial H^i}{\partial t} + \mathcal{R}_H^i + \sigma^{*i} H^i + \nabla E^3 \cdot (\mathbf{e}^3 \times \mathbf{e}^i) + E^3 \mathbf{e}^i \cdot (\nabla \times \mathbf{e}^3) = 0,$$

$$\varepsilon^3 \frac{\partial E^3}{\partial t} + \mathcal{R}_E + \sigma^3 E^3 - \sum_{m=1}^2 \left[\nabla H^m \cdot (\mathbf{e}^m \times \mathbf{e}^3) - H^m \mathbf{e}^3 \cdot (\nabla \times \mathbf{e}^m) \right] = 0,$$

where

$$\mathcal{R}_H^i = \sum_{m=1}^2 \mu^m H^m \left(\frac{\partial \mathbf{e}^m}{\partial t} \cdot \mathbf{e}^i \right), \qquad \mathcal{R}_E = \varepsilon^3 E^3 \left(\mathbf{e}^3 \cdot \frac{\partial \mathbf{e}^3}{\partial t} \right).$$

Weak formulation in moving frames coordinates

the weak form of the above equations is obtained for the test function φ as follows:

$$\int \mu^i \frac{\partial H^i}{\partial t} \varphi dx + \int \sigma^{*i} H^i dx - \int E^3 \nabla \varphi \cdot \mathbf{e}^{3i} dx + \int E^3 \mathbf{e}^3 \cdot (\nabla \times \mathbf{e}^i) \varphi dx + \int_{\partial \mathcal{M}} \mathbf{n} \cdot \left(\mathbf{E}^* \times \mathbf{e}^i \right) \varphi ds = 0,$$

$$\int \varepsilon^3 \frac{\partial E^3}{\partial t} \varphi dx + \int \sigma^3 E^3 dx - \sum_{m=1}^{2} \int H^m \nabla \varphi \cdot \mathbf{e}^{3m} dx - \sum_{m=1}^{2} \int H^m \mathbf{e}^m \cdot \left(\nabla \times \mathbf{e}^3 \right) \varphi dx - \int_{\partial \mathcal{M}} \mathbf{n} \cdot \left(\mathbf{H}^* \times \mathbf{e}^3 \right) \varphi ds = 0,$$

- $\mathbf{n}$ is the edge normal vector.
- $\mathbf{E}^*$ and $\mathbf{H}^*$ are numerical fluxes.

Numerical flux

The solution of Rankine-Hugoniot equations leads to the following upwind flux for TM mode: for $i = 1,\ 2$,

$$-\mathbf{e}^i\cdot(\mathbf{n}\times\mathbf{E}^*)=\mathbf{e}^i\cdot\frac{\left(-\mathbf{n}\times\mathbf{e}^3\right)\{\{Y_i E^3\}\}+0.5\alpha\mathbf{n}\times(\mathbf{n}\times[\mathbf{H}])}{\{\{Y_i\}\}},$$

$$\mathbf{e}^3\cdot(\mathbf{n}\times\mathbf{H}^*)=\mathbf{e}^3\cdot\frac{\sum_{m=1}^{2} n^m\mathbf{e}^m\times\{\{Z_m\mathbf{H}\}\}-0.5\alpha[E^3]}{\sum_{m=1}^{2}\{\{Z_m\}\}}.$$

- $Z_i^{\pm}\equiv\sqrt{\mu^{3-i}/\varepsilon^3}\equiv(Y_i^{\pm})^{-1}$
- $\{\{A\}\}\equiv 0.5(A^{+}+A^{-})$
- $[A]\equiv A^{-}-A^{+}$
- α is in the range of $0<\alpha\leq 1$

For $\alpha=0$ and $Z_i=Y_i=1$, the above flux is reduced to the central flux.

Spectral Convergence

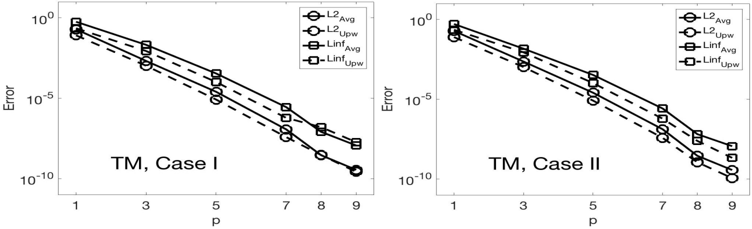


Figure: Error convergence of the test case

- TM mode with PEC boundary conditions:

  $\mathbf{H}=\frac{\pi}{\omega}\sin\omega t[-\mathbf{x}\sin\pi x\cos\pi y+\mathbf{y}\cos\pi x\sin\pi y]$, $\mathbf{E}=\mathbf{z}(\sin\pi x\sin\pi y\cos\omega t)$
- Test Case I): $\mu^t=4.0, \quad \mu^r=0.5,$
- Test Case II): $\mu^t=0.5, \quad \mu^r=(1-2x^2)$

PMLs

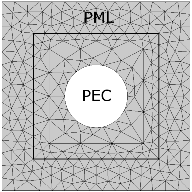


Figure: Right rectangular PML

A perfectly matched layer (PML) is an artificial absorbing layer commonly used to truncate computational domain for simulations with open boundary. The incident wave upon a PML from interior medium does not reflect at the interface and is absorbed strongly.

Oblique PMLs

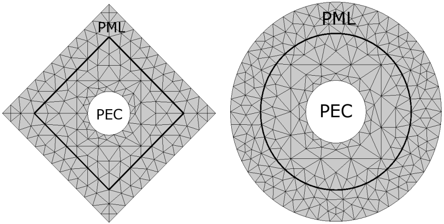


Figure: Oblique rectangular PML and circular PML

PMLs have been applied only to rectangular regions in which boundaries are parallel to the Cartesian coordinate axis. However, moving frames can be adapted to a PML with an oblique angle. However, PMLs for a curved boundary such as the circular PML is still impossible by assumption of planar waves.

PMLs

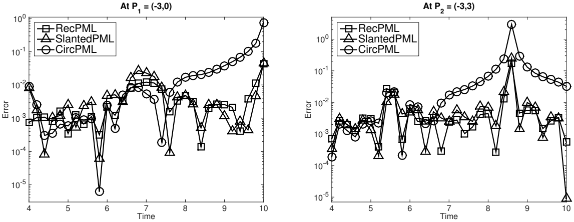


Figure: Relative error of H_z for rectangular PML, oblique PML, and circular PML.

- The observation points are $P_1 = (-3, 0)$ and $P_2 = (-3, 3)$.
- The incident plane waves applied.
- The scatterer is circular PEC.

Invisible Cloak

The mechanism of the metamaterial invisible cloak is based on the annihilation of the field component along the direction of the surface, often analogously expressed as *hiding under the carpet*. The tangential component of the electric field detours the energy flux without significant losses in nontrivial shadow behind the object.

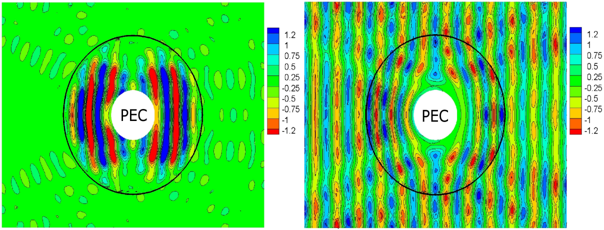


Figure: Scattering of H_z: Distribution of the scattered and total field

- Rectangular PMLs are used.
- Material properties of cloak: $\hat{\mu} = \hat{I}$ and $\hat{\varepsilon} = \varepsilon_t \hat{t} + \varepsilon_r \hat{r}$ where $\varepsilon_t = \left(\frac{b}{b-a}\right)^2$ and $\varepsilon_r = \left(\frac{b}{b-a}\right)^2 \left(\frac{r-a}{r}\right)^2$
- The scattering by the PEC object is significantly reduced by the cloak.

ELF propagation

Extremely low frequency electromagnetic wave propagation propagation (ELF) in the earth-ionosphere waveguide can be modeled as a two-dimensional wave propagation on a curved surface by ignoring the vertical velocity components.
The MMF-scheme should not be influenced by the range of frequency the number of grid points for sufficient grid resolution to reduce the discretization error.

ELF propagation

Extremely low frequency electromagnetic wave propagation propagation (ELF) in the earth-ionosphere waveguide can be modeled as a two-dimensional wave propagation on a curved surface by ignoring the vertical velocity components.
The MMF-scheme should not be influenced by the range of frequency the number of grid points for sufficient grid resolution to reduce the discretization error.
ELF is simulated on various types of surfaces such as sphere irregular or non-convex surfaces.
The wave of higher frequency requires relatively larger domain, but it is more like to be absorrerd by lossy materials such as earth and seawater.

ELF propagation

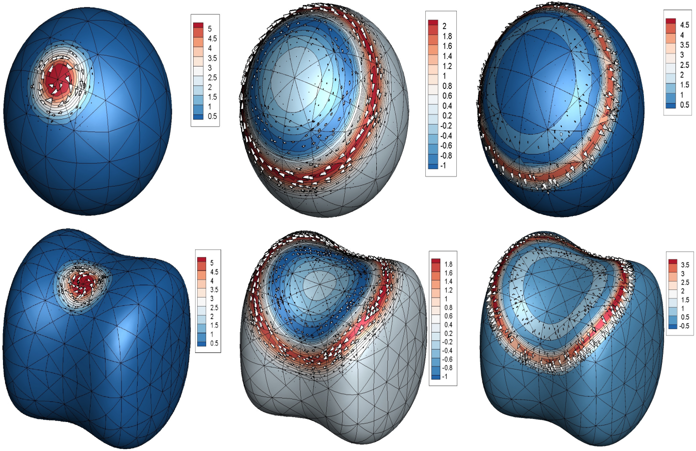
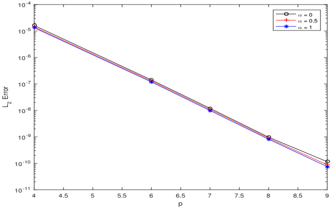


Figure: Electromagnetic field distribution E_z (contour) and $\mathbf{H}$ (arrow) at T $=$ 0.2 (left), T $=$ 1.0 (middle). Energy (contour), and Energy density flux (arrow) (right).

Extensions to 3D



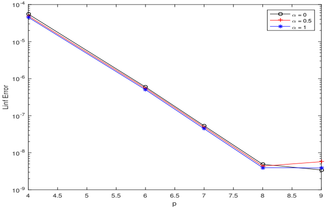
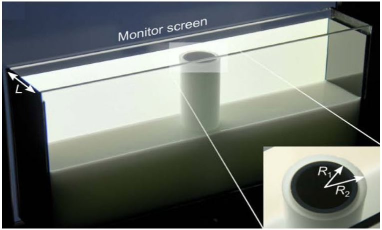


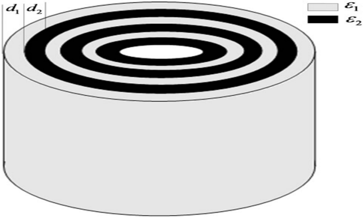
Figure: Error convergence of the test case

- $H^x(x,y,z,t) = \sin(m\pi x)\cos(n\pi y)\cos(j\pi z)\sin(\omega t)$,
  $H^y(x,y,z,t) = \cos(m\pi x)\sin(n\pi y)\cos(j\pi z)\sin(\omega t)$,
  $H^z(x,y,z,t) = \cos(m\pi x)\cos(n\pi y)\sin(j\pi z)\sin(\omega t)$,
- $E^x(x,y,z,t) = \cos(m\pi x)\sin(n\pi y)\sin(j\pi z)\sin(\omega t)$,
  $E^y(x,y,z,t) = \sin(m\pi x)\cos(n\pi y)\sin(j\pi z)\sin(\omega t)$,
  $E^z(x,y,z,t) = \sin(m\pi x)\sin(n\pi y)\cos(j\pi z)\sin(\omega t)$.

With $m = n = j = \omega = 1$.

Three-Dimensional Invisibility Cloak





Conclusions and future directions

- The derivation of upwind flux for anisotropic materials in Maxwell's equations for DG methods.
- The advantage in solving numerically Maxwell's equations on curved surfaces without the metric tensor and composite meshes.
- Extension to 3D formulation is demanding.

Thank You!