Parametric Model Order Reduction with Nektar++

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    - Greedy Sampling
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Projection-Based Parametric Model Order Reduction

Consider linear PDE depending affinely on parameter μ assembled as

$$A(\mu)x(\mu) = b(\mu),$$

- $A(\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) A^q$,
- $b(\mu) = \sum_{q=1}^{Q_b} \Theta_b^q(\mu) b^q$.

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Collect snapshots at different parameter locations $\{\mu_1, \dots, \mu_N\}$ and form a snapshot space [Rozza et al., 2008]

$$V_N = \{x(\mu_1), \dots, x(\mu_N)\}.$$

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$$V_N = \{x(\mu_1), \ldots, x(\mu_N)\}.$$

The space V_N forms a reduced order space, i.e., solutions at intermediate parameter values μ are sought in V_N.

Reduced Order Model

Use snapshot space as ansatz space, i.e., solve

$$V_N^T A(\mu) V_N x_N(\mu) = V_N^T b(\mu),$$

with reduced order solution $x_N(\mu)$ and reduced order system matrix

$$A_N(\mu) = V_N^T A(\mu) V_N = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) V_N^T A^q V_N,$$

i.e., the affine form carries over to the reduced order setting.
Before parameter evaluations with the reduced order model, precompute and store

$$A_N^q = V_N^T A^q V_N,$$

for computational efficiency.

Reduced Basis Concept

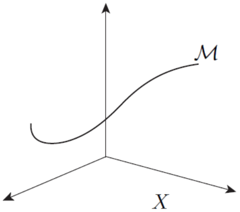

Figure: Solution manifold $\mathcal{M}$ in the discretization space X.

- Find linear space approximating $\mathcal{M} = \{x(\mu)|\mu \in \mathcal{D}\}$
- In many cases a low order space V_N spanned by snapshots recovers $\mathcal{M}$
- Affine form allows an Offline-Online decomposition for computational efficiency

Sampling

Greedy Sampling

- Let Ξ denote a finite sample of parameter domain $\mathcal{D}$.
- Set $S_1 = \{\mu_1\}$ and $V_1 = span\{x(\mu_1)\}$.
- For $N = 2, ..., N_{max}$, find $\mu_N = \arg\max_{\mu \in \Xi} \Delta_{N-1}(\mu)$,
- then set $S_N = S_{N-1} \cup \mu_N, \quad V_N = V_{N-1} + span\{x(\mu_N)\}$.

A simple choice for $\Delta_N(\mu)$ is the residual dual norm

$$\Delta_N(\mu) = \|A(\mu)x_N(\mu) - b(\mu)\|_{X'}.$$

Also rigorous error estimation is possible, involving the parametrized coercivity or inf-sup stability constant.

POD of Time-Trajectory

Given a time trajectory $\{x(t_1), \ldots, x(t_N)\}$ with $\text{span}\{x(t_1), \ldots, x(t_N)\} = X_{N_s}$, a reduced model is generated from the POD modes corresponding to most of the energy of the time trajectory, [Haasdonk and Ohlberger, 2008].
The POD computes the pojection space V_N^{POD} as

$$V_N^{POD} = \arg \inf_{X_N \subset X_{N_s}} \frac{1}{N} \sum_{N=1}^{N_s} \|x(t_i) - \Pi_{X_N}(x(t_i))\|_{L^2}^2 .$$

Numerically, a singular value decomposition is computed as

$$U^T S V = X_{N_s},$$

where S is a diagonal matrix containing the singular values in decreasing order and U conatins the POD modes.
Typically, modes corresponding to 99% of the energy are taken into the reduced order model, which accurately recovers the time trajectory.

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Computational Fluid Dynamics

Incompressible **Navier-Stokes Equations** with velocity u, pressure p, Reynolds number Re and forcing f

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \frac{1}{\text{Re}} \nabla^2 u + f,$$
$$\nabla \cdot u = 0.$$

Considering the Reynolds number as parameter, an affine structure is given.

In case of geometric (non-affine) parameter-dependence, an affine form is approximated $\Rightarrow$ Empirical Interpolation

The nonlinear term is then also projected onto the reduced space, [Lassila et al., 2013].

Example

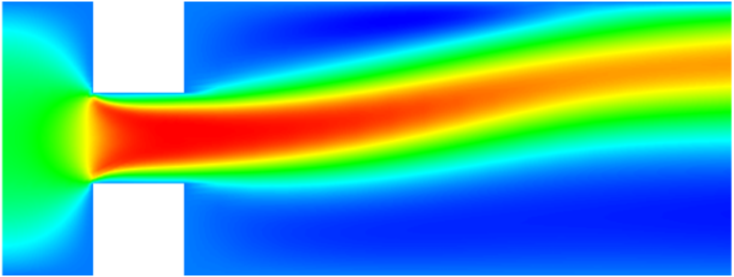

Figure: Channel flow with pitchfork bifurcation.

Research Perspectives:
Recover trajectories with a reduced order model for different Reynolds numbers.
Bifurcation analysis with a reduced model.

Singular Value Decay

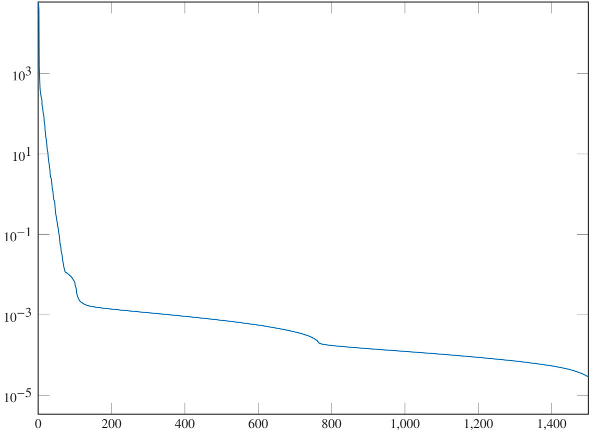

Figure: Singular values of the time trajectory.

A reduced order model of size 11 captures the trajectory of originally 6’939 dofs.

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Representing a Reduced Basis

LocalToGLobal example map

$$\hat{u}_l = \begin{bmatrix} \hat{u}_0^1 \\ \hat{u}_1^1 \\ \hat{u}_0^2 \\ \hat{u}_1^2 \\ \hat{u}_0^3 \\ \hat{u}_1^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix}$$

LocalToGLobal map of a reduced basis would be dense with enries in $\mathbb{R}$

$$\hat{u}_l = \begin{bmatrix} \hat{u}_0^1 \\ \hat{u}_1^1 \\ \hat{u}_0^2 \\ \hat{u}_1^2 \\ \hat{u}_0^3 \\ \hat{u}_1^3 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix}$$

Defining a Parameteric Problem

- Reduced Basis - Driver
- Reduced Basis - Filters

Possibly, a reproduction of parts of the class structure for the reduced order setting is necessary.

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Thank you for your attention!

Acknowledgement:
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References

Model Reduction Overview: www.modelreduction.org

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