Method of moving frames at Nektar++: Current and Future

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About me in Nektar++ Community

- I was a Spencer’s postoc a long time ago.
- But, **NOT** a fluid dynamist nor **NOT** a serious computational programmer.
- Proudly, **a theorist** in geometry, biophysics, electrodynamics.
- My version of Nektar++ is at least one or two years old. So, the new version of Nektar++ does not mean much to me.
- Using Nektar++ for more than 8 years, but still don’t want to know much about something ‘push’, ’gitlab’, ‘boost’, and ‘C11’.
- So I need an handy help to learn the new functions and features of Nektar++.
- Hopefully, these kinds of researchers like me also find Nektar++ to be **convenient** and **useful**.

What is the method of moving frames (MMF) ?

- A continuous and finite group theory on geometry, mainly developed by the famous mathematician, **Élie Cartan** in 1920's.
- '**Moving Frames**' is the English translation of French word '**repére mobile**'. It means <u>moving reference</u> or <u>*moving axis*</u>, not moving mesh.
- 'The method of moving frames' series papers from 2013 is an adaptation of Cartan's unique perspective on geometry for **scientific computing**.

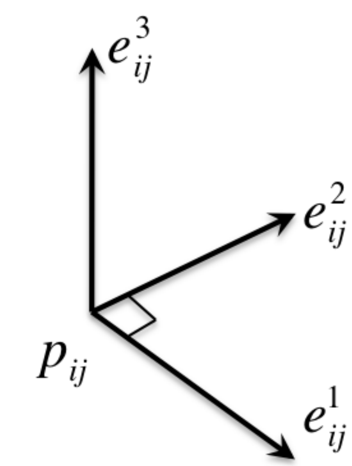
Cartan's perspectives

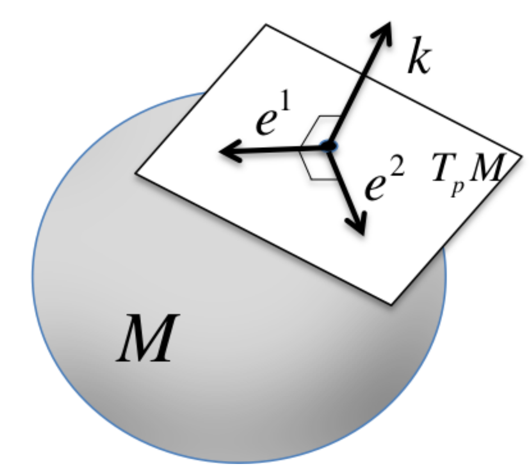
- Geometers say that there is **NO Euclidean or isotropic object** in this world. Everything is curved and has various material properties, as geometry is often defined in **Riemannian sense**.

- Cartan says that every point of curved object has a **infinitesimal** Euclidean coordinate system. So, this means that everything in this world is '**a kind of' isotropic Euclidean object**.

- **This Euclidean axis** is called **moving frames**.

- '**Curved**' and '**anisotropy**' means something only in the **absolute and global coordinate system.**





But, the trick is in <u>infinitesimal</u>

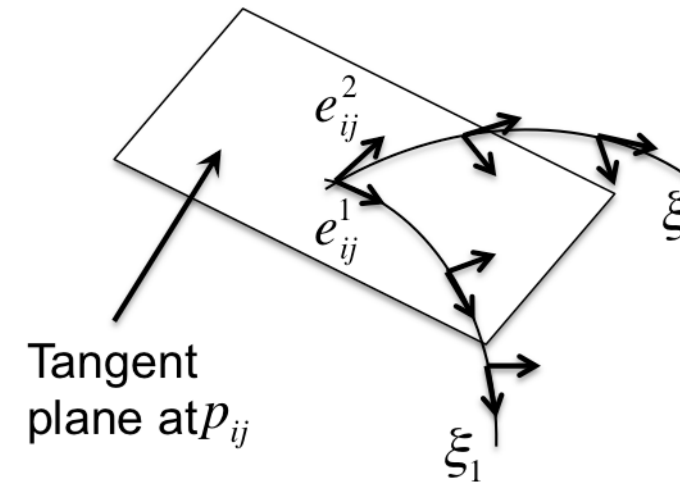
- Velocity of the particle at each grid point is now measured in the **Euclidean reference frame (moving frame) attached to each grid point**.

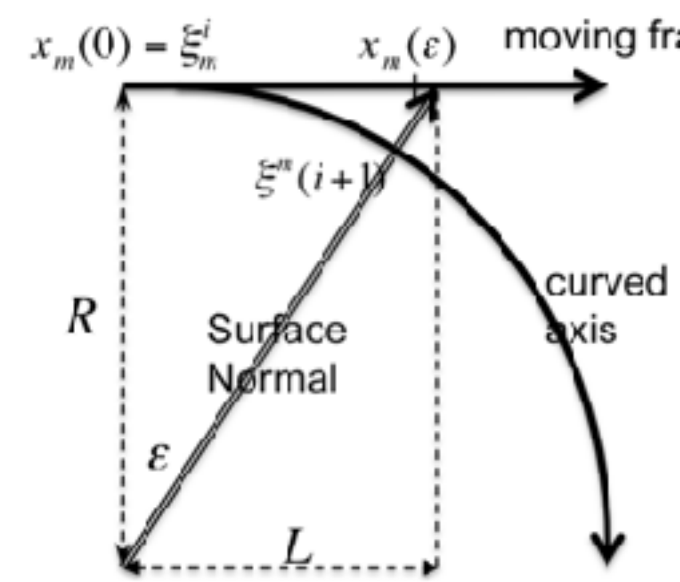
- However, 'Infinitesimal' restriction **cannot be applied in principle** to the discrete world of scientific computing.

- Computational Error = Temporal Error + Spatial Error+ MMF Error (non-infinitesimal on a curve)

- Infinitesimal condition is relieved because MMF error is always a **higher order** of *dx* than the spatial error if p >1.

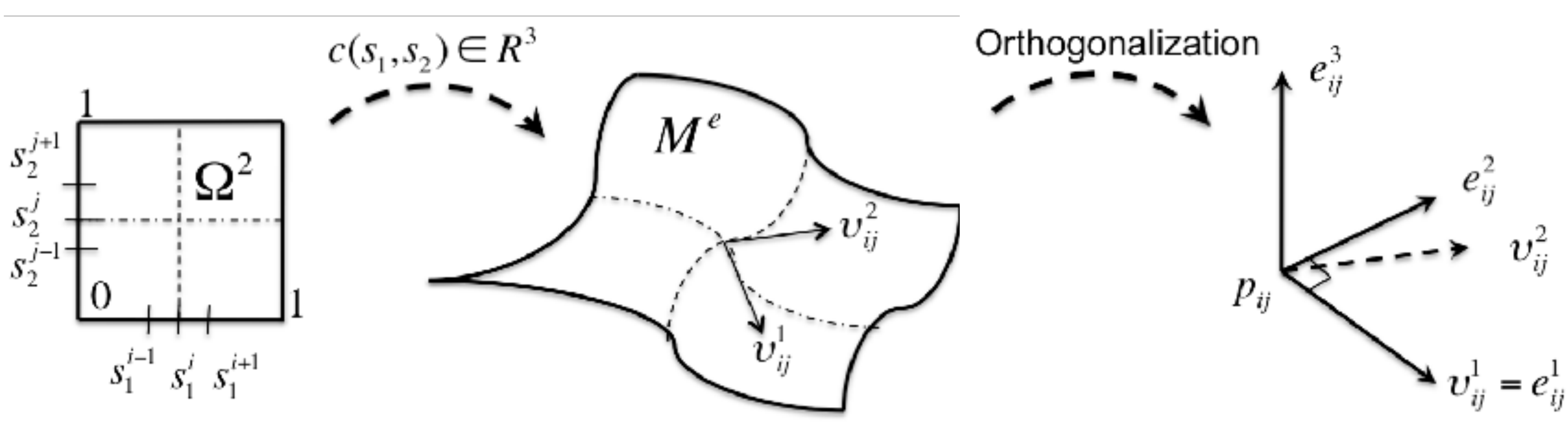
- Computational Error = Temporal Error + Spatial Error dominating the MMF Error, if p >1





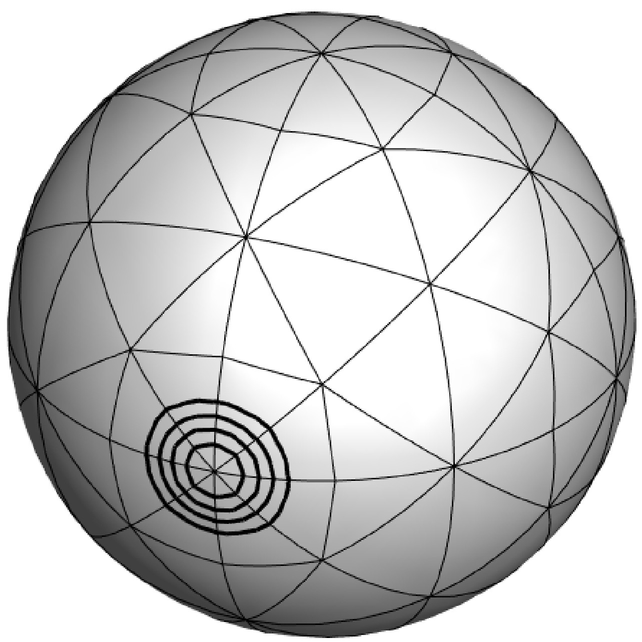
All about the MMF scheme

1. Construct orthonormal vectors at each grid point



2. Expand vector or the gradient in moving frames

1. Method of Moving Frames for Conservational Laws



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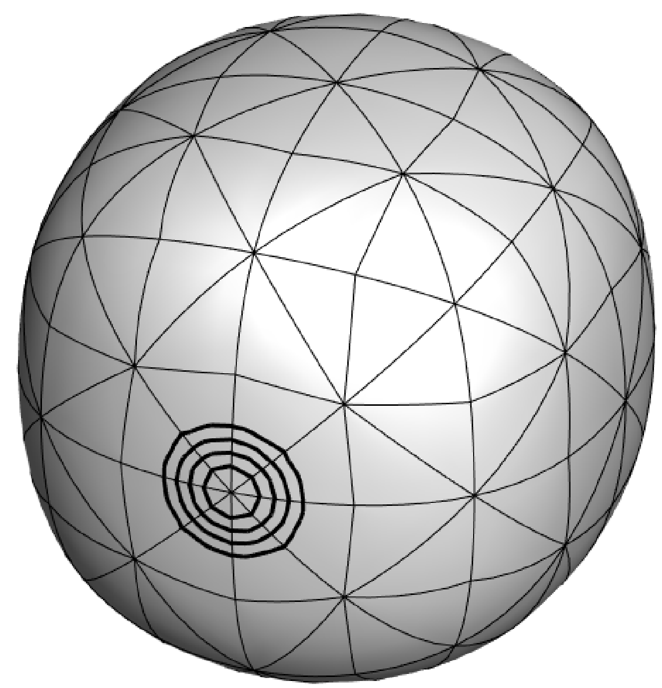
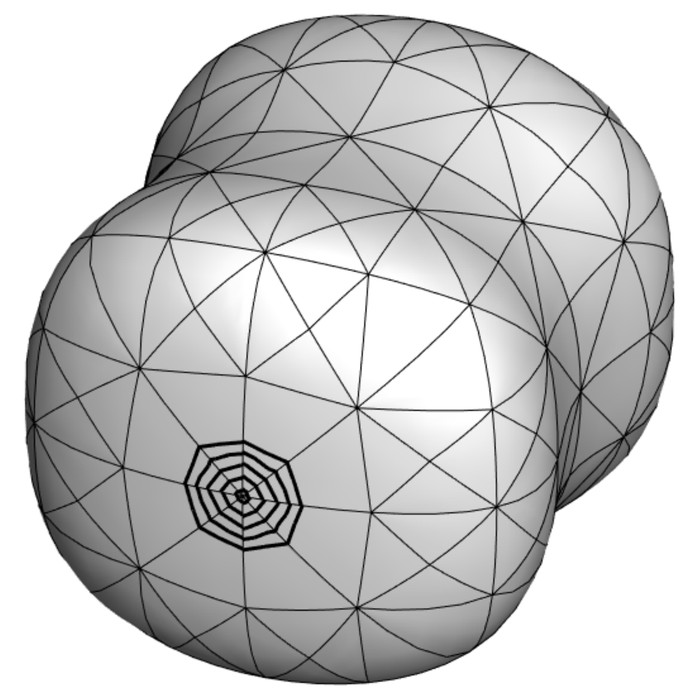


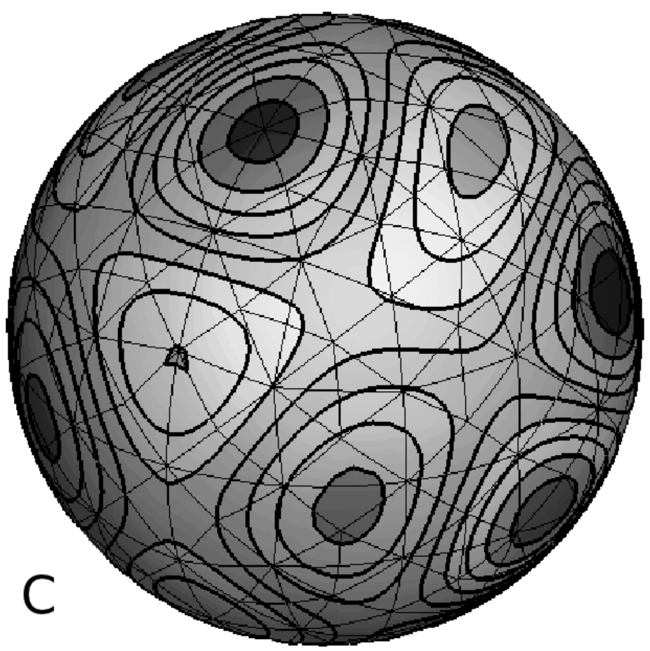
Irregular



Non-convex

Method of moving frames to solve conservation laws on curved surfaces, J. Sci. Compt., 53(2), 268-294, 2012

2. Method of Moving Frames for Anisotropic Diffusion



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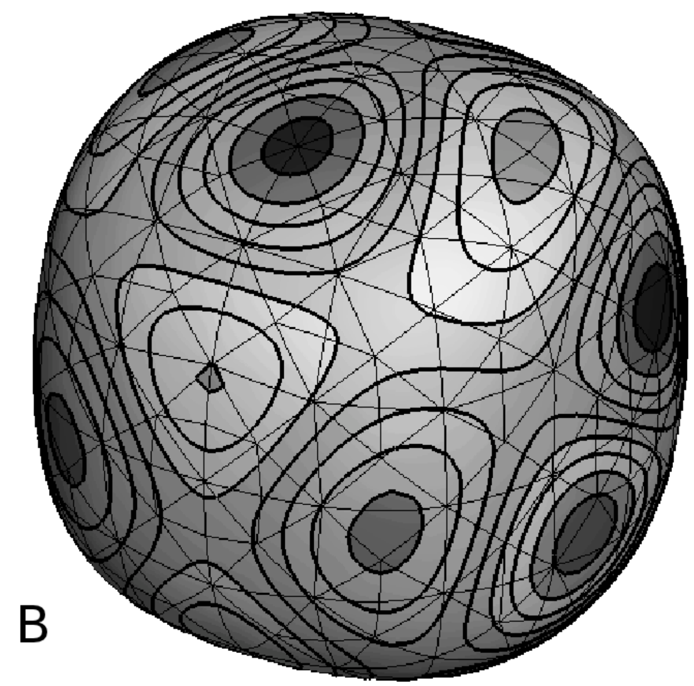

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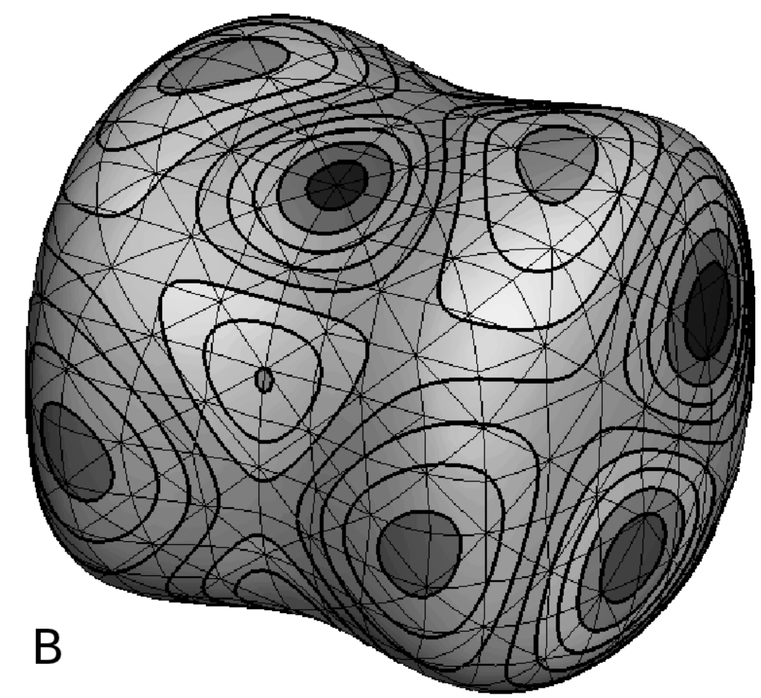
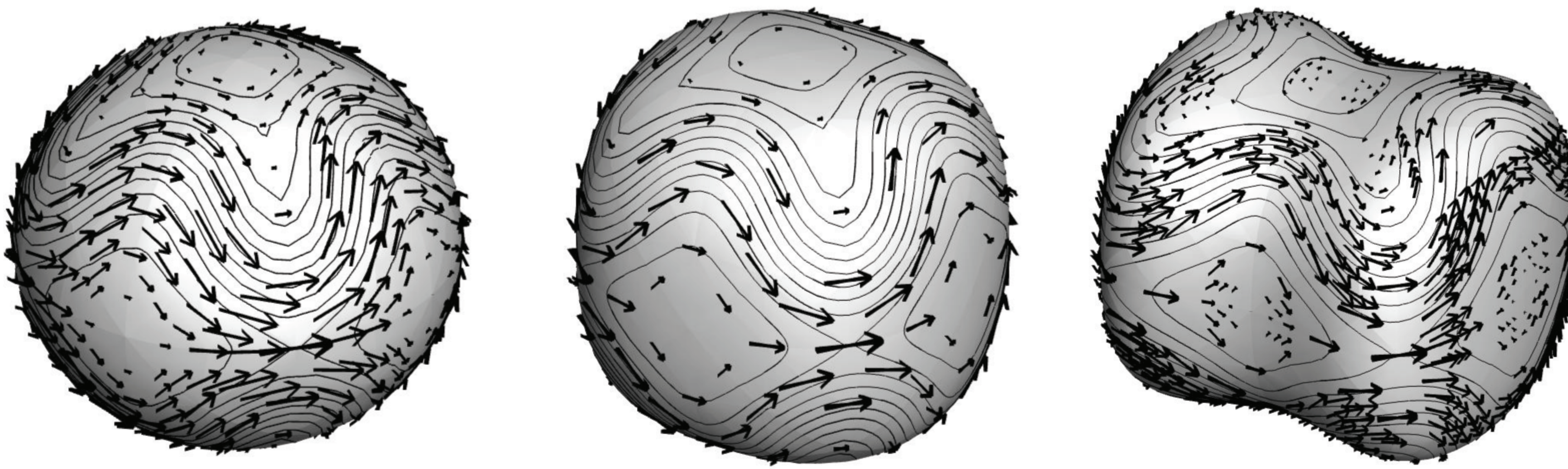


Non-convex

Sehun Chun, Method of moving frames to solve (an)isotropic diffusion equations on curved surfaces, J. Sci. Compt., 59(3), 626-666, 2014.

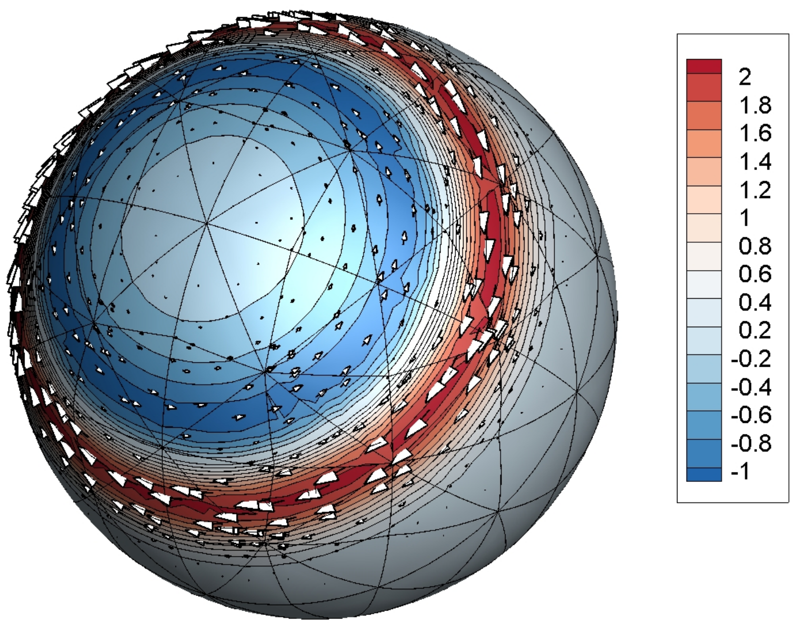
3. Method of Moving Frames for Shallow Water Equations



Sphere Irregular Non-convex

S. Chun and C. Eskilsson, Method of moving frames to solve the shallow water equations on arbitrary rotating curved surfaces, J. Compt. Phys., 333, 1-23, 2017

4. Method of Moving Frames for Maxwell’s equations



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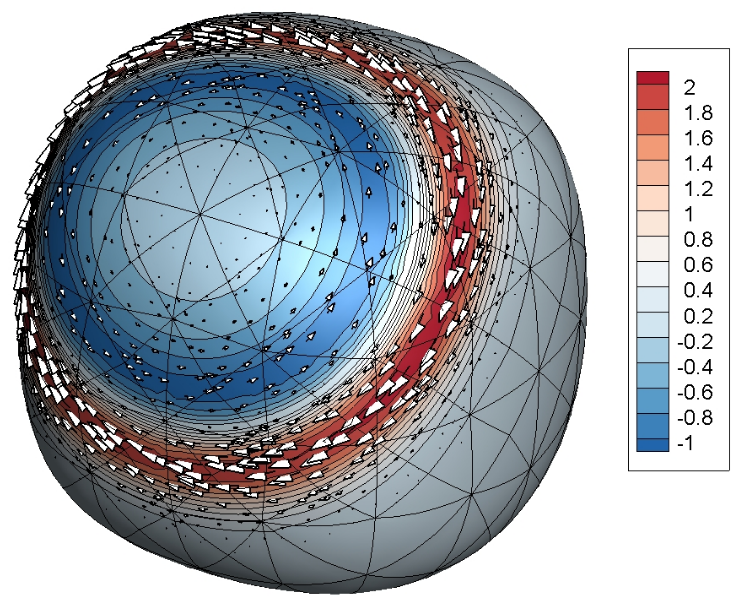

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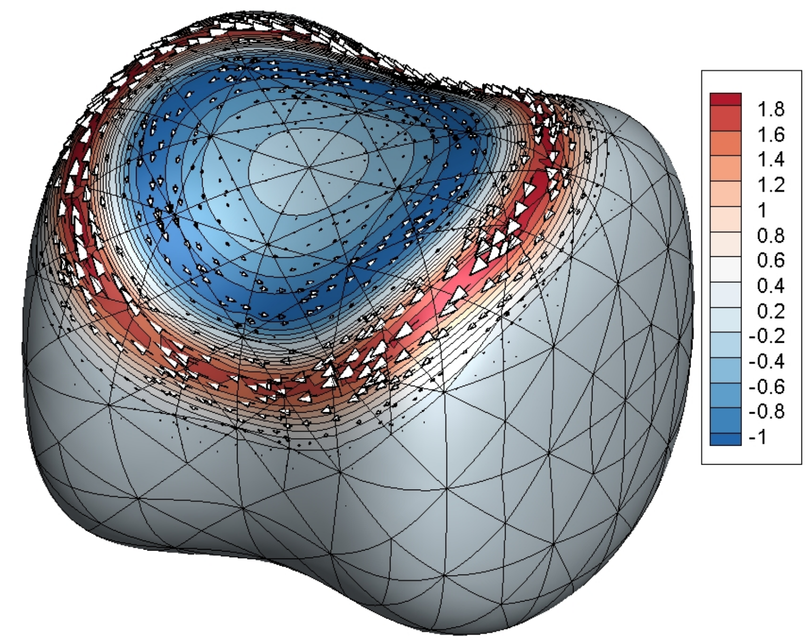


Non-convex

Sehun Chun, Method of moving frames to solve the time-dependent Maxwell's equations on anisotropic curved surfaces: Applications to invisible cloak and ELF propagation, J. Comput. Physics., 340, 85-104, 2017

But, current acceptance is poor

(Two years ago in his/her letter of rejection, an anonymous reviewer wrote…)

In summary, the authors have failed miserably to convert me to the religion of moving frames. However, the writing is much clearer than usual with the exceptions noted above, and the novelty of calculating flows using moving frames for surfaces has merit in that I've never seen any other attempts to do this. The authors deserve a chance to put their ideas before the world. Perhaps the reader will someday find a use for the authors' technology.

Now available at Nektar++

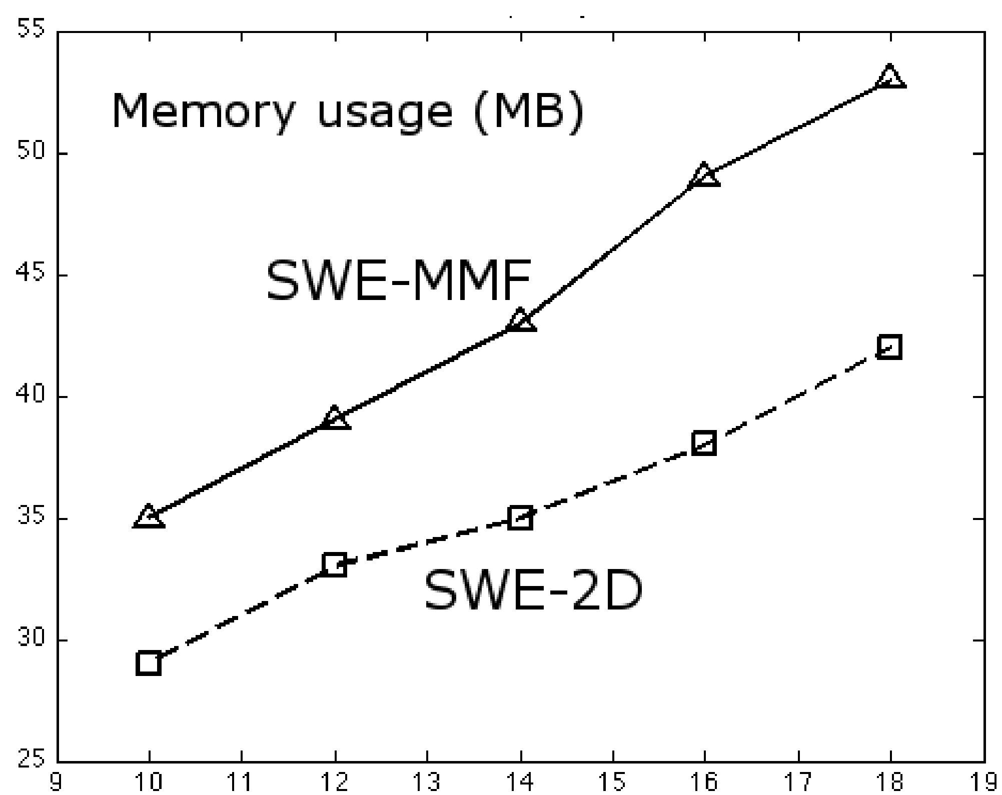
- Incorporation of the **DG-MMF schemes** into Nektar++ has been recently completed.

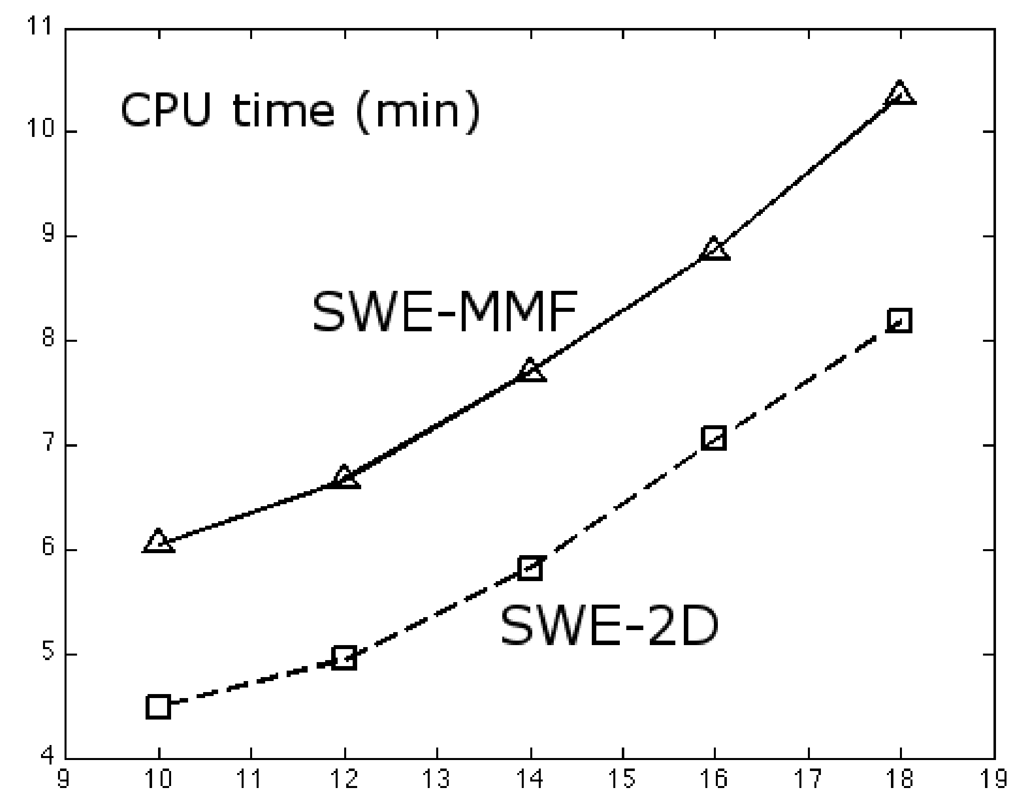
- In Nektar++, we can solve some PDEs on any curved surfaces such as 1) conservational laws, 2) diffusion equations, 3) the SWE, 4) Maxwell's equations.

- Other PDEs can be solved in the very similar ways.

Feature 1: A **2D** scheme with no **surrounding space**

Benefit: Best **scalability** and **least** computational costs





Feature 2: No **metric tensor** or its derivatives

Benefit: Remove **inaccurate** coefficients of the geometric tensor

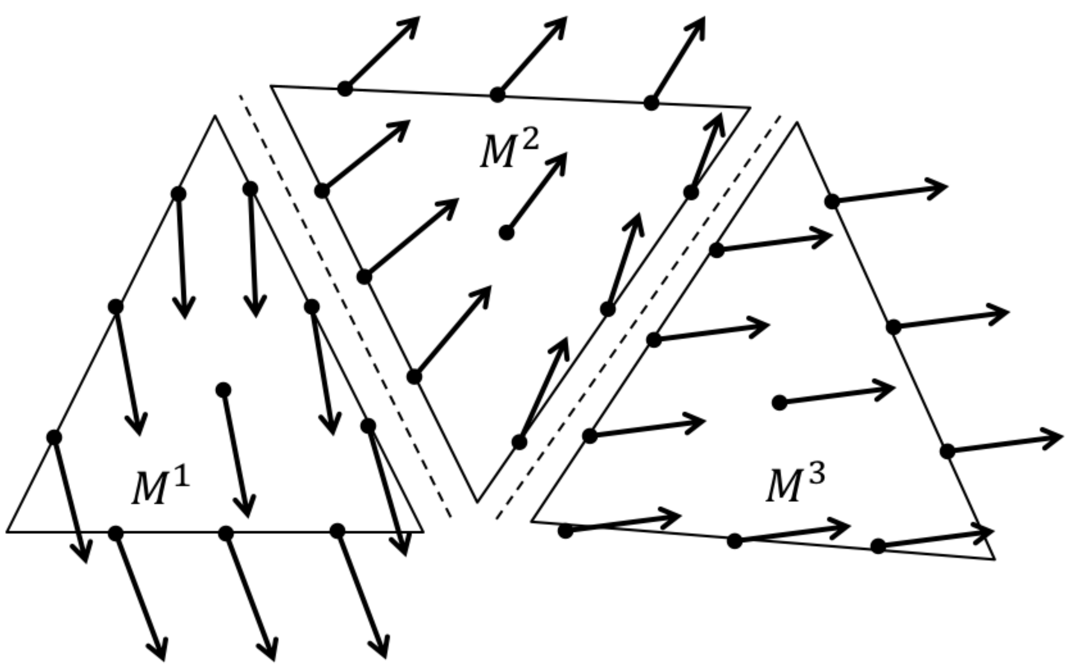
$$\mathbf{V} = \sum_{i=1}^{3} V_u^i \mathbf{u}^i \ (\text{Covariant}) \qquad \mathbf{V} = \sum_{i=1}^{3} V^i \mathbf{e}^i \ (\text{MF})$$

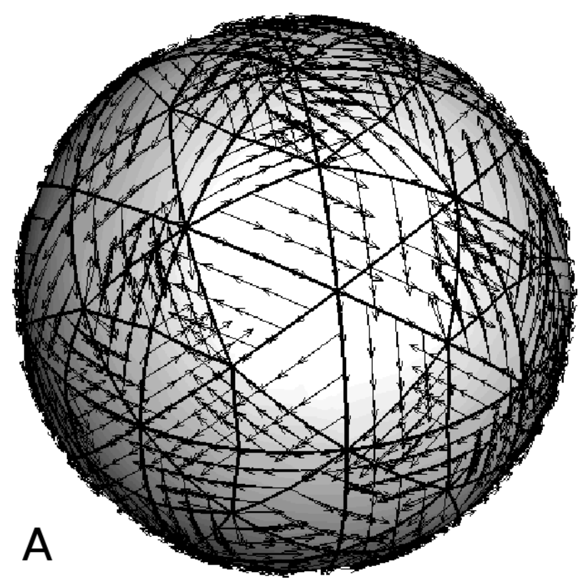
$$\frac{\partial V^i}{\partial \xi_i} = \sqrt{g_{ii}} \left(\frac{\partial V_u^i}{\partial \xi_i} + \Gamma_{ii}^i V_u^i \right)$$

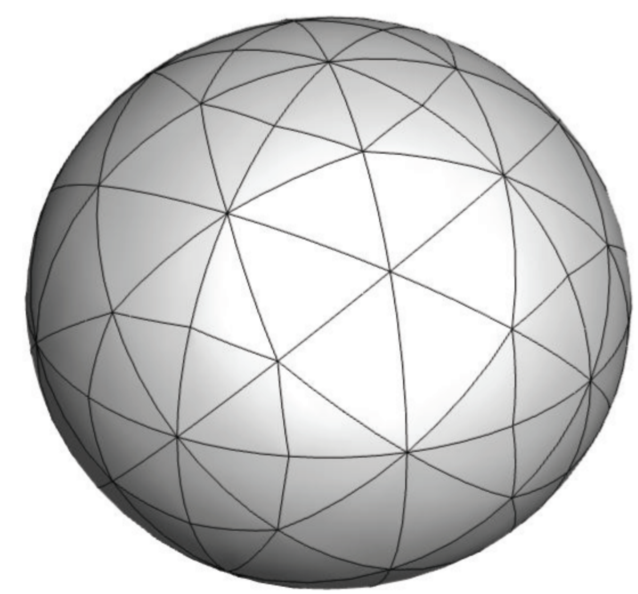
The metric tensor means direct differentiation of the space, but MMF allows the **weak solution** of the same equation

Feature 3: No geometric **singularity**

Benefit: Flexible meshing and adaptivity



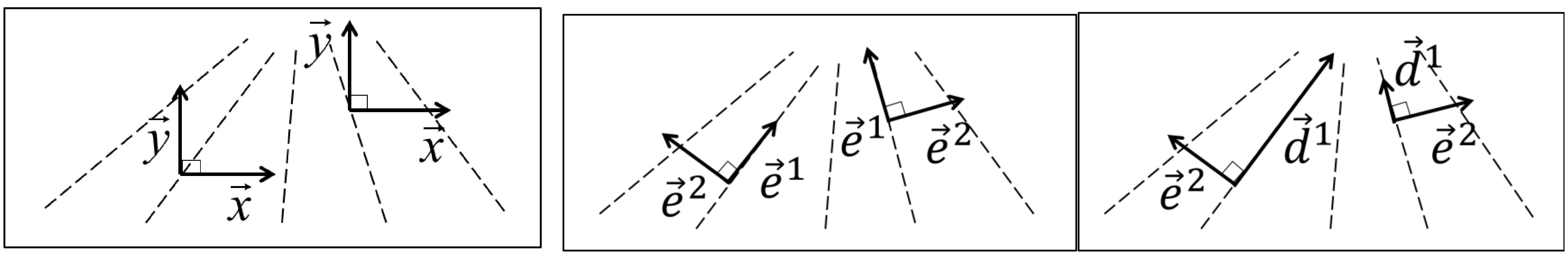




- In covariant formulation, a random axis cannot be chosen because estimating the **curvature of the curved axis is very challenging**.
- In MMF, no need to derive curvature because it is **of unit length** all the time.

Feature 4: Easy applicability to **anisotropy**

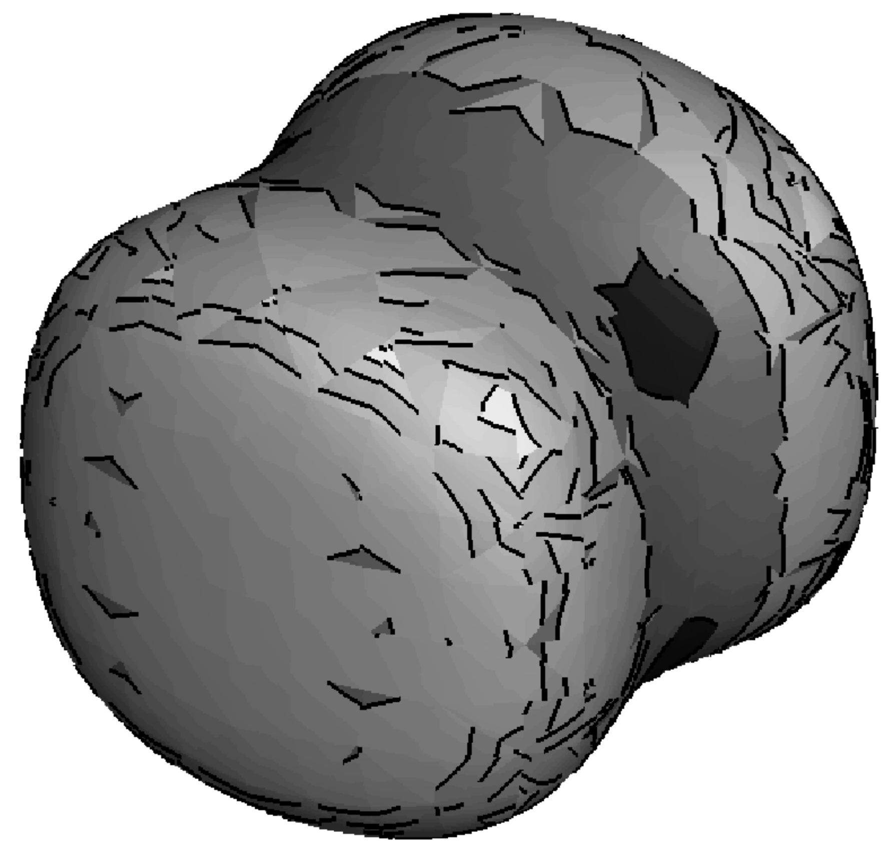
Benefit: Same scheme for isotropy and off-diagonal anisotropy



Off-diagonal anisotropy (left) can be turn into **axial anisotropy** by changing the direction of moving frames (middle), even into **isotropy** by changing the length of the moving frames (right)

Feature 5: for the general surface

Benefit: Applicable to any anisotropic curved surfaces



Distribution of Gaussian curvature for irregular (left) and non-convex surface (right)

Limitations

- It works only for high-order methods (p>1). Maybe MMF does **not** work for **low-order FD or FV** schemes.

- Curved element should have **slowly-varying curvature**. Refinements of the highly-varying curvature can solve this problems.

- Moving frames (**three vectors**) should be **stored** at every grid point.

Near future work

- MMF with NekMesh (CAD definition of surface geometry) for curved surface to **reduce the mesh error of grid points for p>1** by Julian Marcon, Michael Turner, Joaquin Peiro, Spencer J. Sherwin (Imperial college).

- Maxwell **3D simulation in anisotropic** media by Ehsan Kazemi.

Long-term future work

- **Magnetohydrodynamics (MHD):** Born to be anisotropic, Related to Cardiac electrophysiology, global electric climate model.

- **Spatial and temporal deformation:** Hot potato, Something should be done by someone, Need to pave roads in Nektar++

- **MMF-Eikonal solver:** Always pops up in every computational sciences, Easy merits in anisotropic propagation, more understanding on geometry.