On implicit LES / under-resolved DNS via spectral element methods

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Introduction & outline

- Implicit LES vs. under-resolved DNS
- Why does it work and how to apply it?
- Understanding the numerics is essential!
- Eigensolution (dispersion-diffusion) analysis
- Upwind DG vs. CG+SVV
- Numerical experiments with Nektar++
- Focusing on accuracy and robustness
Eigensolution analysis for DG – linear advection in 1D
Eigensolution analysis for DG – the 1% rule
Eigensolution analysis for DG – the 1% rule
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Eigensolution analysis for DG – the 1% rule

LINEAR ADVECTION

<table>
<thead>
<tr>
<th>p</th>
<th>m</th>
<th>(kh)_{1%}^{ld}</th>
<th>2π/(kh)_{1%}^{ld}</th>
<th>(Δ/h)_{1%}^{ld}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1.127</td>
<td>11.150</td>
<td>2.788</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2.616</td>
<td>7.205</td>
<td>1.201</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4.330</td>
<td>5.804</td>
<td>0.726</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6.164</td>
<td>5.097</td>
<td>0.510</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>8.071</td>
<td>4.671</td>
<td>0.389</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>10.027</td>
<td>4.386</td>
<td>0.313</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>12.018</td>
<td>4.183</td>
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<tr>
<td>8</td>
<td>9</td>
<td>14.035</td>
<td>4.029</td>
<td>0.224</td>
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( std. upwinding )
Eigensolution analysis for DG – tests in Burgers turbul.

\[ \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = \frac{A_F}{\sqrt{\Delta t}} \sum_{N \in N_F} \frac{\sigma_N(t)}{\sqrt{|N|}} \exp \left( i \frac{2\pi N}{L} x \right) \]

\[ p = 1, n_{el} = 2048 \quad \text{and} \quad p = 7, n_{el} = 512 \]
Eigensolution analysis for DG – tests in Burgers turbul.

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<tr>
<th>p</th>
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\[
\hat{k}^{1\%} = \frac{(kh)^{1d}}{h} = \frac{12.018}{2/512} \approx 3077
\]

\[
\Rightarrow \log_{10} \hat{k}^{1\%} \approx 3.49
\]
Eigensolution analysis for DG – tests in Burgers turbul.

\[
\begin{array}{cccc}
 p & m & (kh)^{1d}_{1\%} & 2\pi/(kh)^{1d}_{1\%} & (\Delta/h)^{1d}_{1\%} \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 6 & 7 & 10.027 & 4.386 & 0.313 \\
 7 & 8 & \textbf{12.018} & 4.183 & 0.261 \\
 8 & 9 & 14.035 & 4.029 & 0.224 \\
\end{array}
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Eigensolution analysis for DG – tests in Burgers turbulence.

MESH REFINEMENT ( p = 4 )
Eigensolution analysis for DG – tests in Burgers turbul.

INCREASING $p$ (fixed DOFs)
Eigensolution analysis for DG – tests in Burgers turbul.

INCREASING $p$ (fixed DOFs)
Eigensolution analysis for CG – insights into SVV
Eigensolution analysis for CG – insights into SVV

spectral vanishing viscosity

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \mu \frac{\partial}{\partial x} \left( Q \star \frac{\partial u}{\partial x} \right) \approx -\mu \sum_{k} k^2 \hat{Q}_k \hat{u}_k \exp(ikx) \]
Eigensolution analysis for CG – insights into SVV

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spectral vanishing viscosity

strictly true for spectral methods
Eigensolution analysis for CG – insights into SVV

spectral vanishing viscosity

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\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \mu \frac{\partial}{\partial x} \left( Q \star \frac{\partial u}{\partial x} \right) \approx -\mu \sum_k k^2 \hat{Q}_k \hat{u}_k \exp(ikx)
\]

KERNEL ENTRIES NORMALLY INCREASE FROM ZERO

REGULAR DIFFUSION RECOVERED WHEN \( Q_k = 1 \) for all \( k \)

strictly true for spectral methods
Eigensolution analysis for CG – advection + diffusion

\[
\begin{align*}
\text{p} &= 3 \\
\text{Pe}^* &= \frac{a\bar{h}}{\mu} = 10 \\
\text{p} &= 4 \\
\text{Pe}^* &= \frac{a\bar{h}}{\mu} = 10
\end{align*}
\]
Eigensolution analysis for CG – advection+diffusion

\[ p = 5 \]
\[ Pe^* = a\frac{h}{\mu} = 10 \]

\[ p = 6 \]
\[ Pe^* = a\frac{h}{\mu} = 10 \]
Eigensolution analysis for CG – advection+SVV

Proposed ‘power kernel’:

\[
\frac{Q_k}{p} = \left(\frac{k}{p}\right)^{p_{svv}}, \quad p_{svv} = \frac{p}{2}
\]
Eigensolution analysis for CG – irregular features

\[ p = 3 \]
\[ Pe^* = a\bar{h}/\mu = 1 \]

\[ p = 3 \]
\[ Pe^* = a\bar{h}/\mu = 0.1 \]
Eigensolution analysis for CG – irregular features

\[ p = 3 \]

\[ Pe^* = \frac{a \bar{h}}{\mu} = 100 \]

\[ p = 3 \]

\[ Pe^* = \frac{a \bar{h}}{\mu} = 1000 \]
Eigensolution analysis for CG – a Péclet-free SVV

Using \( \mu \propto \frac{ah}{P} \implies \text{fixed } Pe^* = \frac{ah}{\mu} \)

(optimized SVV kernel to mimic DG)
Numerical experiments with Nektar++ (inviscid TGV)
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Conclusions & outlook

- Why does SEM-based iLES/uDNS work?
- How to follow this approach?
- 1% rule as estimate of "implicit" filter width
- Favor high-order with coarser meshes
- Stabilizing techniques at high Reynolds
- Avoid simplistic Riemann fluxes with DG
- Employ well-behaved SVV operators with CG
Questions

